Prep Course Mathematics

Elementary algebra

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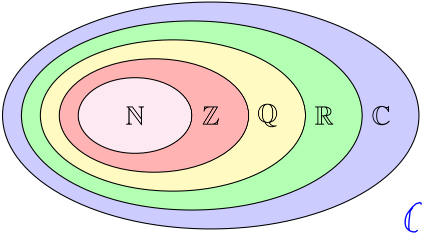
Numbers

You already know the following numbers:

- $\mathbb{N}$ (natural numbers) $1, 2, 3, \ldots$

  $\mathbb{N}_0$ (natural numbers with 0) $0, 1, 2, 3, \ldots$
- $\mathbb{Z}$ (integers) $\ldots -3, -2, -1, 0, 1, 2, 3, \ldots$
- $\mathbb{Q}$ (rational numbers) $\frac{p}{q}$, where p is an integer and q a natural number
- $\mathbb{R}$ (real numbers)

Through elementary operations (addition, subtraction, multiplication and division) two numbers yield another number.



$\mathbb{C}$ complex numbers

	$\mathbb{N}$	$\mathbb{N}_0$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$
0		✓	✓	✓	✓
$\sqrt{\frac{4}{9}}$				✓	✓
$\sqrt{9}$	✓	✓	✓	✓	✓

$$\sqrt{\frac{4}{9}} = \frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}$$

$$\sqrt{9} = 3$$

$0{,}222121212 \ldots$ is rational $\qquad 0{,}4444 = 0{,}\overline{4} \qquad \frac{4}{9}$

$0{,}12112211122211112222\ldots$ is irrational

$0{,}123456789101112\ldots$ is irrational

$0{,}121122111222111113579$ is rational

Conventions

Order of operations:

- Brackets first (from inside to out)
- Exponents
- Division and Multiplication (from left to right)
- Addition and Subtraction (from left to right)

$(3^2)^3 = 9^3 = 729$

$3^{(2^3)} = 3^8 = 6561$

$2 + 2 \cdot 3^2 = 2 + 2 \cdot 9 = 2 + 18 = 20$

⚠ There are some cases where brackets are implied by the notation:

- The fraction line denotes an operation that requires brackets, e.g.
  - $\dfrac{a \pm b}{c \pm d} = \dfrac{(a \pm b)}{(c \pm d)}$
- An exponent itself is always in brackets, e.g.
  - $a^{x \pm y} = a^{(x \pm y)}$
  - $a^{x \cdot y} = a^{(x \cdot y)}$
  - $a^{\frac{x}{y}} = a^{(\frac{x}{y})}$

Brackets

- To override operator precedence, brackets must be used:

$$1 + 3 \cdot 5 = 1 + 15 = 16$$

$$(1+3) \cdot 5 = 4 \cdot 5 = 20$$

- For simplifying a minus in front of brackets, the identity $-(\ldots) = -1 \cdot (\ldots)$ is useful.

$$\begin{aligned} 6 - (3a + 2) &= 6 + (-1) \cdot (3a + 2) \\ &= 6 + (-3a - 2) \\ &= 6 - 3a - 2 \\ &= 4 - 3a \end{aligned}$$

Rules

- **Commutative law:**
  Summands or factors can be arbitrarily interchanged, i.e.

$$a + b = b + a \quad \text{and} \quad a \cdot b = b \cdot a.$$

$3+2 = 2+3$ $\qquad 17 \cdot \frac{1}{2} \cdot x^2 \cdot \pi = 17 \, \frac{1}{2} \cdot \pi \cdot x^2$

$3-2 \neq 2-3$! $\qquad = 17 \cdot \pi \cdot \frac{1}{2} \cdot x^2$

- **Associative law:** $3 + x + (-2) + 7 = 3 + 7 + (-2) + x$
  For summands or factors, order does not matter, i.e.

$$(a + b) + c = a + (b + c) \quad \text{and} \quad (a \cdot b) \cdot c = a \cdot (b \cdot c).$$

$(3+4)+5 = 3+(4+5)$ $\qquad (2 \cdot 3) \cdot 5 = 2 \cdot (3 \cdot 5)$

- **Distributive law:**
  Multiplication and addition combine as

$$a \cdot (b + c) = a \cdot b + a \cdot c.$$

$21 = 3 \cdot 7 = 3 \cdot (5+2) = 3 \cdot 5 + 3 \cdot 2 = 15 + 6 = 21$ $\qquad 2 \cdot (3+x) = 2 \cdot 3 + 2 \cdot x = 2x + 6$

✓ $2^{3^2} = 512$

✗ $2 \cdot 3^2 = 36$ $\quad 2 \cdot 9 = 18$

✓ $3 \cdot 3^3 = 81$

✗ $3 + 2^3 = 125$ $\quad 3 + 8 = 11$

✗ $-2(4-6) = -8 \overset{+}{-} 12 = 4$

✓ $ab(4 - 2ba) = 4ab - 2(ab)^2$

Calculating with fractions

$\frac{p}{q}$ $\leftarrow$ numerator, $\leftarrow$ denominator

The following identities hold:

- Expansion/Simplifying: $\dfrac{a}{b} = \dfrac{a \cdot c}{b \cdot c}$ $\quad c \neq 0$

  $\frac{1}{2} = \frac{1 \cdot 2}{2 \cdot 2} = \frac{2}{4}$ $\qquad \frac{x^2+5x+4}{10x^3+20} = \frac{(x^2+5x+4)\cdot x}{(10x^3+20)\cdot x}$

- Addition/Subtraction: $\dfrac{a}{b} \pm \dfrac{c}{d} = \dfrac{ad \pm bc}{bd}$

  $\frac{2}{5} + \frac{3}{4} = \frac{2 \cdot 4}{5 \cdot 4} + \frac{3 \cdot 5}{4\,5} = \frac{8+15}{20} = \frac{23}{20}$

- Multiplication/Division: $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$ and $\dfrac{\frac{a}{b}}{\frac{c}{d}} = \dfrac{a}{b} \cdot \dfrac{d}{c} = \dfrac{ad}{bc}$

  $\frac{3}{4} \cdot \frac{2}{3} = \frac{6}{12} = \frac{1 \cdot \cancel{6}}{2 \cdot \cancel{6}} = \frac{1}{2}$ $\qquad \frac{\frac{13}{15}}{\frac{2}{7}} = \frac{13}{15} \cdot \frac{7}{2} = \frac{91}{30}$

  $\frac{\cancel{3}}{\cancel{2} \cdot 2} \cdot \frac{\cancel{2}}{\cancel{3}} = \frac{1}{2}$

a) $\frac{5}{6} - \frac{1}{4}\,\frac{2}{3} = \frac{5}{6} - \frac{1}{6} = \frac{5-1}{6} = \frac{4}{6} = \frac{\cancel{2}\cdot 2}{\cancel{2}\cdot 3} = \frac{2}{3}$

$\frac{1}{4}\cdot\frac{2}{3} = \frac{1}{2\cdot\cancel{2}}\cdot\frac{\cancel{2}}{3} = \frac{1}{6}$

b) $\frac{\left(\frac{1}{x}+\frac{1}{x^2}\right)}{\frac{1}{x}} = \frac{x+1}{x\cdot\cancel{x}}\cdot\frac{\cancel{x}}{1} = \frac{x+1}{x} = \frac{x}{x}+\frac{1}{x} = 1+\frac{1}{x}$

$\frac{1}{x}+\frac{1}{x^2} = \frac{1\cdot x}{x\cdot x}+\frac{1}{x^2} = \frac{x+1}{x^2}$

c) $\left(\frac{a}{2}+\frac{b}{4}\right)\left(\frac{1}{\left(\frac{a}{2}\right)}+\frac{2}{a}\right) = \frac{2a+b}{\cancel{4}}\cdot\frac{\cancel{4}}{a} = \frac{2a+b}{a} = \frac{2a}{a}+\frac{b}{a} = 2+\frac{b}{a}$

$\frac{a}{2}+\frac{b}{4} = \frac{2\cdot a}{4}+\frac{b}{4} = \frac{2a+b}{4}$

$\frac{1}{\left(\frac{a}{2}\right)}+\frac{2}{a} = 1\cdot\frac{2}{a}+\frac{2}{a} = \frac{4}{a}$

Binomial expansion

$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

The following identities hold:

1. $(a+b)^2 = a^2 + 2ab + b^2$
2. $(a-b)^2 = a^2 - 2ab + b^2$
3. $(a+b)(a-b) = a^2 - b^2$

$2^2 - 3^2 = (2-3)(2+3)$

$(x+1)^2 - (x-1)^2 = (x^2+2x+1) - (x^2-2x+1) = \cancel{x^2} + 2x \cancel{+1} \cancel{-x^2} + 2x \cancel{-1}$

$= 4x$

Pascals's Dreieck

$(a+2b)^3 = a^3 + 3\cdot a^2(2b) + 3a(2b)^2 + (2b)^3$

$(a+b)^4$

$$
\begin{array}{lccccccccc}
n=0: & & & & & 1 & & & & \\
n=1: & & & & 1 & & 1 & & & \\
n=2: & & & 1 & & 2 & & 1 & & \\
n=3: & & 1 & & 3 & & 3 & & 1 & \\
n=4: & 1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

$a^2 + 2ab + b^2$

$(a+b)^3$

$a^3 + 3a^2b + 3ab^2 + b^3$

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

Deriviation of the 1. binomial formula

$$(a+b)^2 = (a+b)(a+b)$$
$$= a^2 + a \cdot b + ba + b^2$$
$$= a^2 + 2ab + b^2$$

Deriviation of the 2. binomial formula

$$(a-b)^2 = (a-b)(a-b)$$
$$= a^2 - ab - ba + b^2$$
$$= a^2 - 2ab + b^2$$

Deriviation of the 3. binomial formula

$$(a+b)(a-b) = a^2 - \cancel{ab} + \cancel{ba} - b^2$$
$$= a^2 - b^2$$

Powers

For a natural number n and a positive real number a,
✎ we write a^n as a shorthand for $\underbrace{a \cdot a \cdot \ldots a}_{n \text{ mal}}$.

More generally (but with a more complicated interpretation), one can define a^p for an arbitrary real number p.

The following identities hold:

1. $(a^p)^q = a^{pq}$ $\quad (a^4)^3 = a^{4\cdot 3} = a^{12}$
2. $(ab)^p = a^p b^p$ $\quad (ab)^3 = a^3 b^3$
3. $\dfrac{a^p}{b^p} = \left(\dfrac{a}{b}\right)^p$ $\quad \left(\dfrac{a^3}{b^3}\right) = \left(\dfrac{a}{b}\right)^3$
4. $a^p a^q = a^{p+q}$ $\quad a^3 \cdot a^4 = a^{3+4} = a^7$
5. $\dfrac{a^p}{a^q} = a^{p-q}$ $\quad \dfrac{a^3}{a^4} = a^{3-4} = a^{-1} = \dfrac{1}{a}$ $\quad \dfrac{\cancel{a}\cdot\cancel{a}\cdot\cancel{a}}{\cancel{a}\cdot\cancel{a}\cdot\cancel{a}\cdot a} = \dfrac{1}{a}$

$a^0 = 1$

$a^{0-1} = \dfrac{a^0}{a^1} = \dfrac{1}{a}$

$(a+b)^n \neq a^n + b^n$

Roots

root-exponent

Roots are special powers: $\sqrt[p]{a} = a^{\frac{1}{p}}$.

radiant

$$\sqrt[q]{\sqrt[p]{a}} = \left(a^{\frac{1}{p}}\right)^{\frac{1}{q}} = a^{\frac{1}{q}\cdot\frac{1}{p}} = a^{\frac{1}{pq}} = \sqrt[pq]{a}$$

⚠ $\sqrt{a}$ is only defined for $a \geqslant 0$
and is also greater than or equal to zero itself.

From the rules for powers we obtain:

1. $\sqrt[q]{\sqrt[p]{a}} = \sqrt[pq]{a}$ $\quad \sqrt[3]{\sqrt[4]{a}} = \sqrt[3\cdot 4]{a} = \sqrt[12]{a}$ $\quad \sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$

2. $\sqrt[p]{ab} = \sqrt[p]{a}\,\sqrt[p]{b}$ $\quad \sqrt{5184} = \sqrt{64\cdot 81} = \sqrt{64}\cdot\sqrt{81} = 8 \quad 9 = 72$

3. $\dfrac{\sqrt[p]{a}}{\sqrt[p]{b}} = \sqrt[p]{\dfrac{a}{b}}$ $\quad \dfrac{\sqrt{9}}{\sqrt{25}} = \sqrt{\dfrac{9}{25}}$

✎ We simply write $\sqrt{a}$ for the square root $\sqrt[2]{a}$. $= a^{\frac{1}{2}}$

Powers and roots combine well as the latter is just a special case of the former:

$$a^{\frac{p}{q}} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p.$$

a) $\sqrt{2\cdot\sqrt[3]{27\cdot 8}} = \sqrt{2\cdot\sqrt[3]{27}\cdot\sqrt[3]{8}} = \sqrt{2\cdot 3\cdot 2}$

$= \sqrt{4\cdot 3} = \sqrt{4}\cdot\sqrt{3}$

$= 2\cdot\sqrt{3}$

$\sqrt{2^2\cdot 3} = \sqrt{2^2}\sqrt{3} = 2\sqrt{3}$

$(2^2)^{\frac{1}{2}} = 2^{\frac{1}{2}\cdot 2} = 2^1 = 2$

b) $a^1\cdot b^2\cdot a^{\frac{1}{3}}\cdot \underbrace{(b^5)^{\frac{1}{4}}}_{b^{\frac{5}{4}}} = a^{1+\frac{1}{3}}\cdot b^{2+\frac{5}{4}} = a^{\frac{3}{3}+\frac{1}{3}}\cdot b^{\frac{2\cdot 4}{4}+\frac{5}{4}} = a^{\frac{4}{3}}\, b^{\frac{13}{4}}$

Calculating percentages

Fundamental identity: $$\frac{\text{percentage}}{100\%} = \frac{\text{percentage value}}{\text{base value}}$$

Computation of a percentage value

What is 5 % of 120 € ?

$$\frac{5}{100} = \frac{x}{120\,€} \Rightarrow x = 120\,€ \cdot \frac{5}{100} = 6\,€$$

Computation of a percentage

What percentage of 120 € is 6€

$$\frac{x}{100} = \frac{6\,€}{120\,€} \Rightarrow x = 100 \cdot \frac{6\,€}{120\,€} = 5\,\%$$

Computation of a base value

150% of the base value is 180 €. What is the base value?

$$\frac{150}{100} = \frac{180\,€}{x} \Rightarrow x = 180\,€ \cdot \frac{100}{150} = 120\,€$$