Analytic geometry and matrices

1. $\cos(\theta) = \frac{\langle\langle \mathbf{a},\mathbf{b}\rangle\rangle}{\|\mathbf{a}\|\cdot\|\mathbf{b}\|} = \frac{-2}{\sqrt{11}\cdot\sqrt{18}} \stackrel{TR}{\Longrightarrow} \theta \approx 98.2°$

2. a) $\left\| \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} \times \begin{pmatrix} 6 \\ 11 \\ 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 11 \\ -6 \\ -8 \end{pmatrix} \right\| = \sqrt{221}$

   b) $\left\| \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 7 \\ -3 \\ 2 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 5 \\ 15 \\ 5 \end{pmatrix} \right\| = 5\sqrt{11}$

3. a) $E = \left\{ \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} : \lambda, \mu \in \mathbb{R} \right\}$

   b) A vector orthogonal to E is $\begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix} \times \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -6 \end{pmatrix}$. A parameter representation of g is thus

$$g = \left\{ \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} + \lambda \cdot \begin{pmatrix} 2 \\ -2 \\ -6 \end{pmatrix} : \lambda \in \mathbb{R} \right\}$$

4. a) A point V (with location vector $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$) lies in E if and only if the connecting vector from P to V, i.e. the vector $\mathbf{v} - \mathbf{p}$, lies in E - that is it is orthogonal to the normal vector $\mathbf{u}$. A normal form and coordinate form of the plane E is therefore

$$\begin{aligned} E &= \left\{ \mathbf{v} \in \mathbb{R}^3 : \langle \mathbf{u}, \mathbf{v} - \mathbf{p} \rangle = 0 \right\} \\ &= \left\{ \mathbf{v} \in \mathbb{R}^3 : \langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{p} \rangle \right\} \\ &= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x + 2y + z = 2 \right\}. \end{aligned}$$

   b) A normal vector of E is

$$\mathbf{n} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix}.$$

Hence a normal form is:

$$E = \left\{ \mathbf{v} \in \mathbb{R}^3 : \left\langle \mathbf{v} - \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} \right\rangle = 0 \right\}.$$

By substituting the location vector $\mathbf{p}$ of P we obtain

$$\left\langle \mathbf{p} - \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} \right\rangle \neq 0.$$

So P does not lie in the plane E. To calculate the distance between P and E, we first note that

$$E = \left\{ \mathbf{v} \in \mathbb{R}^3 : \left\langle \mathbf{v} - \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\rangle = 0 \right\}$$

is a Hesse normal form of E. Hence the distance is

$$\left| \left\langle \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\rangle \right| = \frac{4}{\sqrt{3}}.$$

5. a) $\alpha = 1; \beta = 1$
   b) $\alpha = -1; \beta = 1$
   c) $\alpha = \frac{1}{3}; \beta = \frac{5}{3}$
   d) $\alpha = 2; \beta = 1$

6. Solving the matrix with Gaussian elimination:

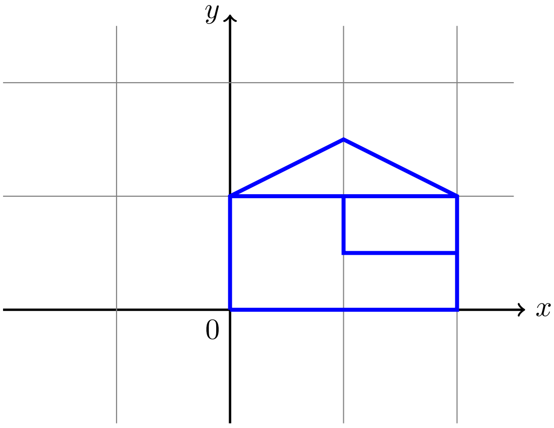
$$\begin{array}{c} I \\ II \\ III \end{array} \quad \left(\begin{array}{ccc|c} 1 & 4 & 3 & 58 \\ 3 & 2 & 1 & 40 \\ 5 & 5 & a & 81 \end{array}\right)$$

$$\begin{array}{c} I \\ II - 3I : IV \\ III - 5I : V \end{array} \quad \left(\begin{array}{ccc|c} 1 & 4 & 3 & 58 \\ 0 & -10 & -8 & -134 \\ 0 & -15 & a-15 & -209 \end{array}\right)$$

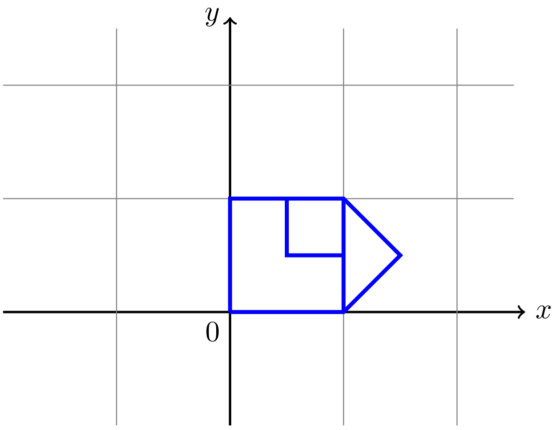
$$\begin{array}{c} I \\ 0,5IV \\ V + 1,5IV : VI \end{array} \quad \left(\begin{array}{ccc|c} 1 & 4 & 3 & 58 \\ 0 & 5 & 4 & 67 \\ 0 & 0 & a-3 & -8 \end{array}\right)$$

From the last line can be inferred that $a = 3$ results in a contradiction. Moreover, we can solve the system of equations for any $a \neq 3$, so there would always be a definite solution.

7. a)



b)



c)