Prep Course Mathematics

Exponential and logarithmic functions

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Exponential functions

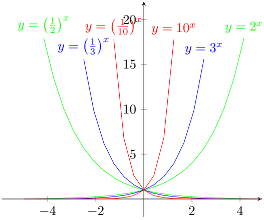
- Power function: variable base, fixed power
  Examples: x^2, $\sqrt{x}$ etc.

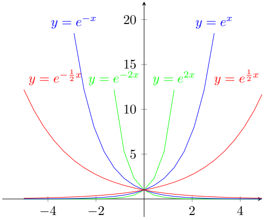
- Exponential function: fixed base, variable power
  Examples: 2^x, $\left(\frac{1}{4}\right)^x$
  In general: a^x where $a > 0$

- The most important base is the Eulerian number $e = 2.718\ldots$ (characterized by the fact that its associated exponential function e^x is its own derivative).

$2^{-x} = \frac{1}{2^x} = \left(\frac{1}{2}\right)^x$

- Rules for powers apply without change.
  Example: $2^{x+3} = 2^x 2^3 = 2^x \cdot 8$





Rules:

- $\exp(x) \cdot \exp(y) = \exp(x+y)$
- $\dfrac{\exp(x)}{\exp(y)} = \exp(x-y)$
- $(\exp(x))^y = \exp(x \cdot y)$

$2^{x+3} = 2^x 2^3$

$(2^3)^4 = 2^{3 \cdot 4}$

Examples:

Find the solution for x without using logarithms

① $3^x - 4 \cdot 3^{x-2} = 15$

$\Leftrightarrow 3^x - 4 \cdot \dfrac{3^x}{3^2} = 15$

$\Leftrightarrow 3^x \left(1 - \dfrac{4}{9}\right) = 15$

$\Leftrightarrow 3^x \left(\dfrac{5}{9}\right) = 15$

$\Leftrightarrow 3^{x} = 27 = 3^{3}$

By comparing exponents, we get $x = 3$

② $(7^x)^{2x-4} = (7^{x+4})^{x-2}$

$\Leftrightarrow 7^{(2x-4) \cdot x} = 7^{(x-2)(x+4)}$

By comparing exponents, we get the following

$(2x-4)x = (x-2)(x+4)$

$\Leftrightarrow 2x^2 - 4x = x^2 + 4x - 2x - 8$

$\Leftrightarrow x^2 - 6x + 8 = 0$

$\Leftrightarrow x_\pm = \dfrac{6}{2} \pm \sqrt{\dfrac{6^2}{4} - 8} = \dfrac{6}{2} \pm \sqrt{\dfrac{36}{4} - \dfrac{32}{4}} = \dfrac{6}{2} \pm \sqrt{1}$

$\Leftrightarrow x_+ = 3 + 1 = 4$ and $x_- = 3 - 1 = 2$

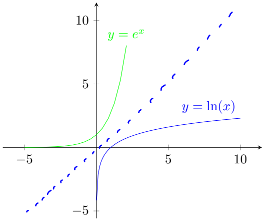
Aim: Represent y as a power of a

Question: How to find the exponent x
for given numbers a and y
such that $a^x = y$

$\Downarrow$

Logarithm for base a

Logarithm



The logarithm $\log_a(x)$ „cancels" the exponentiation a^x, i.e.

$$\log_a(a^x) = x \quad \text{and} \quad a^{\log_a(x)} = x.$$

For base $a = e$, the function $\log_e(x)$ has the shorthand $\ln(x)$.

Important rules

$$\log_a(x) + \log_a(y) = \log_a(xy)$$

$$\log_a(x) - \log_a(y) = \log_a\left(\frac{x}{y}\right)$$

$$\log_a(x^p) = p\log_a(x)$$

$\ln(2 \cdot 2) = \ln(2) + \ln(2)$

$\ln(2 \cdot 2) = \ln(2^2) = 2 \cdot \ln(2)$

Change of base

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$$

Examples: $\log_b(b^x) = x \underbrace{\log_b(b)}_{1} = x$

$\log_{10}(10000) = \log_{10}(10^4) = 4$

$\log_2(1024) = \log_2(2^{10}) = 10$

$\log_5(\sqrt{5}) = \log_5(5^{\frac{1}{2}}) = \frac{1}{2}$

$\log_{10}\left(\frac{1}{100}\right) = \log_{10}\left(\frac{1}{10^2}\right) = \log_{10}(10^{-2}) = -2$

$\ln(e^5) = 5$

$e^{\ln(4)} = 4$

$10^{\log_{10}(4)} = 4$

Change of base: $\log_b(x) = \dfrac{\log_a(x)}{\log_a(b)}$

Examples: $\log_{10}(e^x) = \dfrac{\log_e(e^x)}{\log_e(10)} = \dfrac{\ln(e^x)}{\ln(10)} = \dfrac{x}{\ln(10)}$

$$\ln(x+2) + \log_{10}(x+2) = \ln(x+2) + \frac{\ln(x+2)}{\ln(10)} = \ln(x+2)\cdot\left(1 + \frac{1}{\ln(10)}\right)$$

Rules for calculating with logarithms

Examples:

- $\log_{10}(20) + \log_{10}(0.5) = \log_{10}(20 \cdot 0{,}5) = \log_{10}(10) = 1$
- $\ln(2e^5) = \ln(2) + \ln(e^5) = \ln(2) + 5 \cdot \underbrace{\ln(e)}_{1} = \ln(2) + 5$
- $\log_{10}(5-x) + \log_{10}(5+x) = \log_{10}((5-x)(5+x)) = \log_{10}(5^2 - x^2) = \log_{10}(25 - x^2)$ $\quad -5 < x < 5$
- $\ln(x^2 - 6x + 9) - \ln(x-3) = \ln\left(\frac{x^2 - 6x + 9}{x-3}\right) = \ln\left(\frac{(x-3)^{\cancel{2}}}{\cancel{x-3}}\right) = \ln(x-3)$ $\quad x > 3$
- $\log_{10}\left(\frac{1}{1000^3}\right) = \log_{10}(1000^{-3}) = -3 \cdot \log_{10}(1000) = -3 \log_{10}(10^3)$

  $= -3 \cdot 3 \cdot \underbrace{\log_{10}(10)}_{=1} = -3 \cdot 3 = -9$
- $\ln(\sqrt{e}^5) = \ln((e^{\frac{1}{2}})^5) = \ln(e^{\frac{5}{2}}) = \frac{5}{2} \underbrace{\ln(e)}_{=1} = \frac{5}{2}$

Equations involving exponential functions or logarithms

To solve equations in which the variable appears in an exponent or a logarithm, the same **approach** as for radical equations is used:

1. Isolate the logarithm or the exponential function in which the variable appears.
2. Cancel the function by taking exponentials or logarithms on both sides.
3. Repeat the first two steps until all exponentials or logarithms have been eliminated.
4. Solve the resulting equation.

Examples for exponential functions

① $e^{2x+3} = 4$

$\Leftrightarrow \ln(e^{2x+3}) = \ln(4)$

$\Leftrightarrow 2x+3 = \ln(4)$

$\Leftrightarrow x = \dfrac{\ln(4)-3}{2}$

② $\left(2^{3-x}\right)^{2-x} = 1$

$\Leftrightarrow 2^{(2-x)(3-x)} = 1$

$\Leftrightarrow \log_2 2^{(2-x)(3-x)} = \log_2(1) = 0$

$\Leftrightarrow (2-x)(3-x) = 0$

$\Leftrightarrow x = 2$ or $x = 3$

③ For $b > 0$, $b \neq 1$

$\sqrt[4]{b^{x-a}} = \sqrt[5]{b^{x+a}}$

$\Leftrightarrow \left(b^{x-a}\right)^{\frac{1}{4}} = \left(b^{x+a}\right)^{\frac{1}{5}}$

$\Leftrightarrow b^{\frac{x-a}{4}} = b^{\frac{x+a}{5}}$

$\Leftrightarrow \ln\left(b^{\frac{x-a}{4}}\right) = \ln\left(b^{\frac{x+a}{5}}\right)$

$\Leftrightarrow \frac{x-a}{4} \cdot \cancel{\ln(b)} = \frac{x+a}{5} \cdot \cancel{\ln(b)}$

$\Leftrightarrow \frac{x-a}{4} = \frac{x+a}{5}$

$\Leftrightarrow 5(x-a) = 4(x+a)$

$\Leftrightarrow 5x - 5a = 4x + 4a$

$x = 9a$

Examples for logarithmic functions

① $\ln\left(\frac{1}{2+x}\right) = 0$ for $x > -2$

$\Leftrightarrow e^{\ln\left(\frac{1}{2+x}\right)} = e^0$

$\Leftrightarrow \frac{1}{2+x} = 1$

$\Leftrightarrow 1 = 2 + x$

$\Leftrightarrow x = -1$

② $\ln(e^x - 5e) = 1$

$\Leftrightarrow e^{\ln(e^x - 5e)} = e^1$

$\Leftrightarrow e^x - 5e = e$

$\Leftrightarrow e^x = 6e$

$\Leftrightarrow \ln(e^x) = \ln(6e)$

$\Leftrightarrow x = \ln(6) + \ln(e) = \ln(6) + 1$

③ For $x > 0$

$\log_{10}(x^3) + 2\log_{10}(x^2) = 21$ $\longrightarrow$ $3 \cdot \log_{10}(x) + 2 \cdot 2\log_{10}(x) = 21$

$\Leftrightarrow 7 \cdot \log_{10}(x) = 21$

$\Leftrightarrow \log_{10}(x^3) + \log_{10}(x^4) = 21$

$\Leftrightarrow \log_{10}(x^3 \cdot x^4) = 21$

$\Leftrightarrow \log_{10}(x^7) = 21$

$\Leftrightarrow 7\log_{10}(x) = 21$

$\Leftrightarrow \log_{10}(x) = 3$

$\Leftrightarrow 10^{\log_{10}(x)} = 10^3$

$\Leftrightarrow x = 10^3 = 1000$