Prep Course Mathematics

Analytic geometry and matrices

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TUHH

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Content

1. Analytic Geometry II
   - Inner product, cross product
   - Orthogonality
   - Representing planes
   - Positional relationships
2. Matrices
   - Basic operations
   - Maps of the form $\mathbf{x} \mapsto \mathbf{A}\mathbf{x}$

Inner product, norm and angle

Standard inner product (Defintion)

Consider any vectors

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

in $\mathbb{R}^n$. Then the (standard) inner product of $\mathbf{v}$ and $\mathbf{w}$ is defined as follows:

$$\langle \mathbf{v}, \mathbf{w} \rangle := v_1 w_1 + v_2 w_2 + \ldots + v_n w_n = \sum_{k=1}^{n} v_k w_k \ .$$

In particular, the length $\|\mathbf{v}\|$ of $\mathbf{v}$ (also called norm) can be written as follows:

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{v_1^2 + v_2^2 + v_3^2 + \ldots + v_n^2}$$

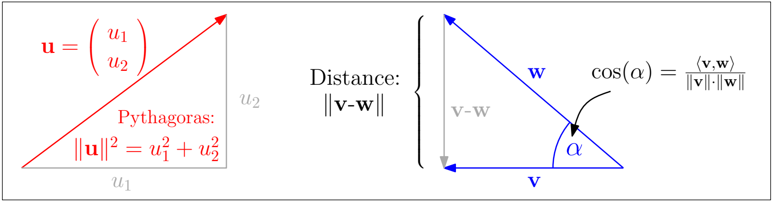
Example: $\left\langle \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -7 \\ 2 \\ 2 \end{pmatrix} \right\rangle = 1 \cdot (-7) + 2 \cdot 2 + 3 \cdot 2 = -7 + 4 + 6 = 3$ $\quad \left\| \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$

Inner product, norm and angle

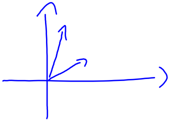
Length, distance, angle

Consider any $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, let $\langle \cdot, \cdot \rangle$ be the standard inner product and let $\| \cdot \|$ denote the norm. Then

- $\|\mathbf{v}\|$ is the length of the vector $\mathbf{v}$,
- $\|\mathbf{v} - \mathbf{w}\|$ is the distance of $\mathbf{v}$ and $\mathbf{w}$,
- the angle α between $\mathbf{v}$ and $\mathbf{w}$ is given by $\cos(\alpha) = \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\|\mathbf{v}\| \cdot \|\mathbf{w}\|}$.



Inner product, norm and angle



Exercise: Consider $\mathbf{v} := \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{w} := \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

(a) Determine the lengths of these vectors.

(b) Determine the angle between $\mathbf{v}$ and $\mathbf{w}$.

a) $\left\| \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\| = \sqrt{2^2+1^2} = \sqrt{5}$ $\qquad \left\| \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\| = \sqrt{1^2+3^2} = \sqrt{10}$

b) $\langle v, w \rangle = \left\langle \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\rangle = 2 \cdot 1 + 1 \cdot 3 = 5$

$\cos(\alpha) = \frac{\langle v, w \rangle}{\|v\| \cdot \|w\|} = \frac{5}{\sqrt{5} \cdot \sqrt{10}} = \frac{\sqrt{25}}{\sqrt{50}} = \sqrt{\frac{25}{50}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$

$\Rightarrow \alpha = \frac{\pi}{4}$ (radian)

$= 45^\circ$ (degree)

Inner product, norm and angle

$$\cos(\alpha) = \frac{\langle v, u_x \rangle}{\|v\| \cdot \|u_x\|}$$

Exercise: Consider $\mathbf{v} := \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{w} := \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

$$0 = \frac{\langle v, u_x \rangle}{\|v\| \cdot \|u_x\|} = \langle v, u_x \rangle$$

(c) Extra problem: For which value $x \in \mathbb{R}$ is the distance between

$$\mathbf{u}_x := \begin{pmatrix} x \\ 1 - x \end{pmatrix}$$

$$0 = \left\langle \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} x \\ 1-x \end{pmatrix} \right\rangle = 2x + 1 - x = x + 1$$

$$x = -1$$

and $\mathbf{v}$ the smallest?

$$\|v - u_x\| = \left\| \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} x \\ 1-x \end{pmatrix} \right\| = \left\| \begin{pmatrix} 2-x \\ x \end{pmatrix} \right\| = \sqrt{\underbrace{(2-x)^2}_{4-4x+x^2} + x^2} = \sqrt{\underline{2x^2 - 4x + 4}}$$

$$=: f(x)$$

Since the root function is a strictly monotonically increasing function, we need to find the minimum of $f(x)$.

$$f(x) = 2x^2 - 4x + 4$$

$$f'(x) = 4x - 4 \stackrel{!}{=} 0 \Rightarrow x = 1$$

$$f''(x) = 4 \qquad f''(1) = 4 > 0$$

$\Rightarrow$ Minimum for $x = 1$

Hence, for $x = 1$ the distance between v and u_x is the smallest.

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \|v - u_1\| = \left\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\| = \sqrt{2}$$

Orthogonality

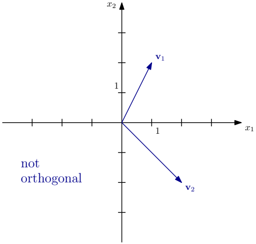
Orthogonality

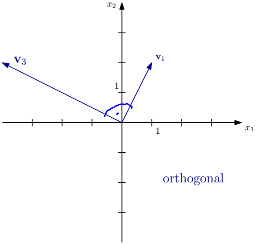
Two vectors $\mathbf{v}, \mathbf{w}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ are orthogonal if and only if $\langle \mathbf{v}, \mathbf{w} \rangle = 0$.

Example: Consider the vectors

$$\mathbf{v}_1 := \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \; \mathbf{v}_2 := \begin{pmatrix} 2 \\ -2 \end{pmatrix} \; \text{ and } \; \mathbf{v}_3 := \begin{pmatrix} -4 \\ 2 \end{pmatrix}.$$

Then $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = -2$ and $\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = 0$. Hence, $\mathbf{v}_1$ and $\mathbf{v}_3$ are orthogonal. But $\mathbf{v}_1$ and $\mathbf{v}_2$ are not orthogonal.





Orthogonality

Orthogonality

Two vectors $\mathbf{v}, \mathbf{w}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ are orthogonal if and only if $\langle \mathbf{v}, \mathbf{w} \rangle = 0$.

Exercise: Consider a triangle with the corner points $A := (-1, 0, 1)$, $B := (1, 2, 3)$ and $C := (-2, 2, 0)$.

(i) Determine the vectors which lead from A to B, from A to C and from B to C.

(ii) Check whether the given triangle is right-angled.

(i) $v_{AB} = \begin{pmatrix} 1-(-1) \\ 2-0 \\ 3-1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ $\quad v_{AC} = \begin{pmatrix} -2-(-1) \\ 2-0 \\ 0-1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$ $\quad v_{BC} = \begin{pmatrix} -2-1 \\ 2-2 \\ 0-3 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ -3 \end{pmatrix}$

(ii) $\langle v_{AB}, v_{AC} \rangle = 2\cdot(-1) + 2\cdot 2 + 2\cdot(-1) = 0$

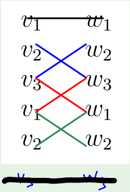
$\Rightarrow$ at A the triangle is right angled.

Cross product

Cross product (Definition)

Let $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ be vectors in $\mathbb{R}^3$. Then the cross product of $\mathbf{v}$ and $\mathbf{w}$ is defined as follows: $\mathbf{v} \times \mathbf{w} := \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$.

Mnemonic



$\rightsquigarrow$ write the vectors next to each other and
below them add the first two components again

$\rightsquigarrow$ delete the first row

$\rightsquigarrow$ determine the entries of $\mathbf{v} \times \mathbf{w}$ using the drawn crosses:

$$\mathbf{v} \times \mathbf{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$$

Cross product

Cross product (Definition)

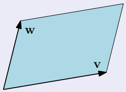
Let $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ be vectors in $\mathbb{R}^3$. Then the cross product of $\mathbf{v}$ and $\mathbf{w}$ is defined as follows: $\mathbf{v} \times \mathbf{w} := \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$.

Important properties

(a) The cross product is only defined in $\mathbb{R}^3$!!!

(b) The vector $\mathbf{v} \times \mathbf{w}$ is orthogonal to $\mathbf{v}$ and $\mathbf{w}$.

(c) The parallelogramm whose sides are given by $\mathbf{v}$ and $\mathbf{w}$ has an area of size $\|\mathbf{v} \times \mathbf{w}\|$.



Exercise

(a) Calculate $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix}$.

$$= \begin{pmatrix} (-2)\cdot(-3) - 1\cdot 3 \\ 1\cdot 0 - 3\cdot(-3) \\ 3\cdot 3 - (-2)\cdot 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \\ 9 \end{pmatrix}$$

$$\begin{array}{cc} \cancel{3} & \cancel{0} \\ -2 & 3 \\ 1 & -3 \\ 3 & 0 \\ -2 & 3 \\ \cancel{1} & \cancel{-3} \end{array}$$

$$\begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix} \times \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -9 \\ -9 \end{pmatrix}$$

Exercise

(b) Consider the vectors

$$\mathbf{v} := \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{w} := \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} .$$

$$\begin{array}{cc} \cancel{1} & \cancel{-1} \\ 2 & 0 \\ 1 & 2 \\ 1 & -1 \\ 2 & 0 \\ \cancel{1} & \cancel{2} \end{array}$$

(i) Find a vector which is orthogonal to $\mathbf{v}$ and $\mathbf{w}$.

(ii) Determine the area of the parallelogramm whose sides are given by the vectors $\mathbf{v}$ and $\mathbf{w}$.

(iii) Find a vector which is orthogonal to $\mathbf{v}$ and $\mathbf{w}$, and the length of which is 1.

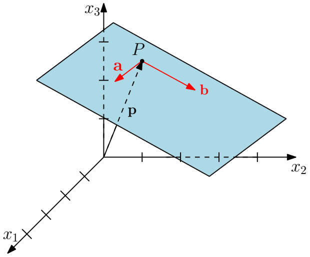
(i) $v \times w = \begin{pmatrix} 2\cdot 2 - 1\cdot 0 \\ 1\cdot(-1) - 1\cdot 2 \\ 1\cdot 0 - 2\cdot(-1) \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$

(ii) $\| v \times w \| = \left\| \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} \right\| = \sqrt{4^2 + (-3)^2 + 2^2} = \sqrt{29}$

(iii) $\frac{1}{\sqrt{29}} \cdot \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$

Representing planes: parameter form

Idea: A plane is determined uniquely if we know a point in the plane and two vectors in the plane which have different directions.



Every point in the plane E can be reached from P by walking along multiples of the vector $\mathbf{a}$ and $\mathbf{b}$.

Hence, every point in E can be represented by a position vector of the form $\mathbf{p}+\lambda\mathbf{a}+\mu\mathbf{b}$

Parameter form of a plane

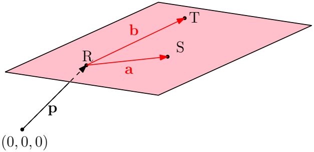
Every plane in $\mathbb{R}^3$ can be described in the following form:

$$E = \{\mathbf{p} + \lambda\mathbf{a} + \mu\mathbf{b} : \ \lambda, \mu \in \mathbb{R}\} = \mathbf{p} + \mathsf{Span}(\mathbf{a}, \mathbf{b}),$$

where the points of E are represented by their position vectors. This representation is called parameter form.

Example

In $\mathbb{R}^3$ there exists a unique plane E which contains the points $R := (1,1,1)$, $S := (1,3,2)$ and $T := (-1,4,3)$. We determine a parameter form of E.



point in the plane: $\mathbf{p} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

vectors in the plane: $\overset{\overset{V_{RS}}{\shortparallel}}{\mathbf{a}} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ and $\overset{\overset{V_{RT}}{\shortparallel}}{\mathbf{b}} = \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix}$

A parameter form of E is:

$$E = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix} \;:\; \lambda, \mu \in \mathbb{R} \right\}$$
$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \text{Span}\left(\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix} \right).$$

Exercise

(a) Find a parameter form of the plane E which contains the following points:

$$A := (6,3,0), \quad B := (-3,10,2) \text{ and } C := (5,3,3)\,.$$

(b) Find a parameter form of the plane F which contains the point $D := (9,1,9)$ and is parallel to the plane E.

Solution: a) point in the plane: $\begin{pmatrix} 6 \\ 3 \\ 0 \end{pmatrix}$

vectors in the plane: $a = v_{AB} = \begin{pmatrix} -3-6 \\ 10-3 \\ 2-0 \end{pmatrix} = \begin{pmatrix} -9 \\ 7 \\ 2 \end{pmatrix}$ $\quad b = v_{AC} = \begin{pmatrix} 5-6 \\ 3-3 \\ 3-0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$

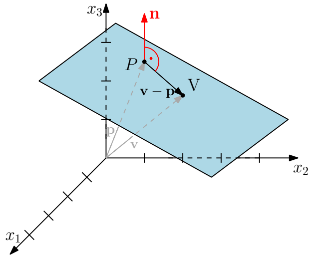
$$E = \left\{ \begin{pmatrix} 6 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -9 \\ 7 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} : \lambda, \mu \in \mathbb{R} \right\} = \begin{pmatrix} 6 \\ 3 \\ 6 \end{pmatrix} + \operatorname{Span}\left(\begin{pmatrix} -9 \\ 7 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} \right)$$

b) point in the plane: $\begin{pmatrix} 9 \\ 1 \\ 9 \end{pmatrix}$

$$F = \left\{ \begin{pmatrix} 9 \\ 1 \\ 9 \end{pmatrix} + \lambda \cdot \begin{pmatrix} -9 \\ 7 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} : \lambda, \mu \in \mathbb{R} \right\} = \begin{pmatrix} 9 \\ 1 \\ 9 \end{pmatrix} + \operatorname{Span}\left(\begin{pmatrix} -9 \\ 7 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} \right)$$

Representing planes: normal form

Idea: A plane E in $\mathbb{R}^3$ is determined uniquely if we know a point in the plane E and a vector (different from $\mathbf{o}$) which is orthogonal to E.



Let $\mathbf{n}$ be a vector that is orthogonal to E.

For every point V on E the vector **v-p** between P and V is orthogonal to $\mathbf{n}$.

Hence, it holds that $\langle \mathbf{v}\text{-}\mathbf{p}, \mathbf{n}\rangle = 0$ for every position vector $\mathbf{v}$ of a point on E.

Normal form of a plane in $\mathbb{R}^3$

Every plane E in $\mathbb{R}^3$ can be described in the following form:

$$E = \left\{\mathbf{v} \in \mathbb{R}^3 : \ \langle \mathbf{v} - \mathbf{p}, \mathbf{n}\rangle = 0\right\} = \left\{\mathbf{v} \in \mathbb{R}^3 : \ \langle \mathbf{v}, \mathbf{n}\rangle = \langle \mathbf{p}, \mathbf{n}\rangle\right\}.$$

Here, $\mathbf{p}$ represents any point of E. The vector $\mathbf{n} \neq \mathbf{o}$ is called normal vector of E, i.e. it is orthogonal to E. This representation is called normal form.

Representing planes: coordinate form

Coordinate form of a plane

Every plane E in $\mathbb{R}^3$ can be written in the form

$$E = \left\{ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : \; a_1x_1 + a_2x_2 + a_3x_3 = d \right\}$$

with $a_1, a_2, a_3, d \in \mathbb{R}$. This representation is called coordinate form.

The equation $a_1x_1 + a_2x_2 + a_3x_3 = d$ describes which condition a point (x_1, x_2, x_3) must satisfy in order to part of the plane.

How to find a coordinate form

Let a plane E be given in normal form $E = \left\{ \mathbf{v} \in \mathbb{R}^3 : \; \langle \mathbf{v} - \mathbf{p}, \mathbf{n} \rangle = 0 \right\}$ then a coordinate equation can be found as follows:

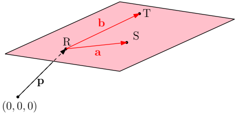
Set $\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and rearrange $\langle \mathbf{v} - \mathbf{p}, \mathbf{n} \rangle = 0$ to get an equation of the form $a_1x_1 + a_2x_2 + a_3x_3 = d$.

Example

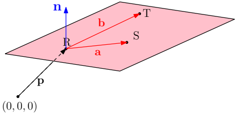
Example: In $\mathbb{R}^3$ there exists a unique plane E which contains the points $R := (1,1,1)$, $S := (1,3,2)$ and $T := (-1,4,3)$.

We already know a parameter form of E:

$$E = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix} \; : \; \lambda, \mu \in \mathbb{R} \right\}$$



For a normal form of E we need a point $\mathbf{p}$ and a normal vector $\mathbf{n}$. (Hint: Use cross product!)



point: $\mathbf{p} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

normal vector: $\mathbf{n} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$

A normal form of E is:

$$E = \left\{ \mathbf{v} \in \mathbb{R}^3 : \left\langle \mathbf{v} - \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{p}, \underbrace{\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}}_{n} \right\rangle = 0 \right\} .$$

Example

Example: In $\mathbb{R}^3$ there exists a unique plane E which contains the points $R := (1,1,1)$, $S := (1,3,2)$ and $T := (-1,4,3)$.

A normal form of E is:

$$E = \left\{ \mathbf{v} \in \mathbb{R}^3 : \left\langle \mathbf{v} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} \right\rangle = 0 \right\} .$$

To obtain a coordinate form, we substitute

$$\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

into the equation of the normal form:

$$\left\langle \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{\begin{pmatrix} x_1 - 1 \\ x_2 - 1 \\ x_3 - 1 \end{pmatrix}}, \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} \right\rangle = 0 \quad \Leftrightarrow \quad 1x_1 - 2x_2 + 4x_3 = 3 \ .$$

Hence,

$$E = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : \ x_1 - 2x_2 + 4x_3 = 3 \right\} .$$

$(x_1 - 1) \cdot 1 + (x_2 - 1)(-2) + (x_3 - 1) \cdot 4 = 0$

$1 \cdot x_1 - 1 - 2x_2 + 2 + 4x_3 - 4 = 0 \qquad | +3$

$1x_1 - 2x_2 + 4x_3 = 3$

Exercise

Find a coordinate form for each of the following planes:

(i) $E_1 := \left\{ \mathbf{v} \in \mathbb{R}^3 : \left\langle \mathbf{v} - \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\rangle = 0 \right\}$

(ii) $E_2 := \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} : \lambda, \mu \in \mathbb{R} \right\}$

Solution

(i) $\left\langle \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\rangle = 0 \iff \left\langle \begin{pmatrix} x_1 - 2 \\ x_2 \\ x_3 - 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\rangle = 0 \iff (x_1 - 2)\, 0 + x_2 \cdot 1 + (x_3 - 1) \cdot 2 = 0$

$\iff x_2 + 2x_3 = 2$

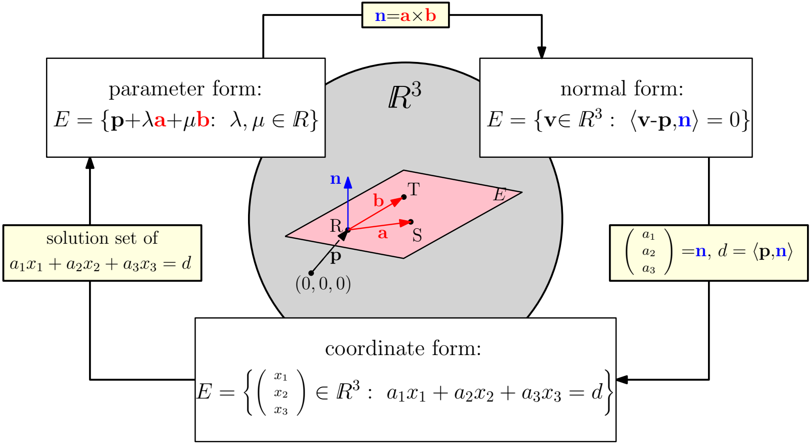
$E_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_2 + 2x_3 = 2 \right\}$

(ii) $n = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ $\quad \begin{array}{cc} \cancel{1} & \cancel{0} \\ 0 & -1 \\ 1 & 1 \\ 1 & 0 \\ 0 & -1 \\ \cancel{1} & \cancel{1} \end{array}$ $\quad \Rightarrow E_2 = \left\{ v \in \mathbb{R}^3 : \left\langle v - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \right\rangle = 0 \right\}$

$\left\langle \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \right\rangle = 0 \iff x_1 + 1 - x_2 + 1 - x_3 = 0 \iff x_1 - x_2 - x_3 = -2$

$E_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_1 - x_2 - x_3 = -2 \right\}$

Overview: representations



Hesse normal form and distance

Hesse normal form (Definition) HNF

A normal form $\left\{\mathbf{v} \in \mathbb{R}^3 : \ \langle \mathbf{v} - \mathbf{p}, \mathbf{n} \rangle = 0\right\}$ of a plane is called Hesse normal form if the length of $\mathbf{n}$ equals 1. $(\|n\| = 1)$

Comment: A normal form can be transformed into a HNF by dividing the given normal vector by its length.

Example: Consider the plane

$$E := \left\{\mathbf{v} \in \mathbb{R}^3 : \ \left\langle \mathbf{v} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\rangle = 0 \right\}.$$

The given representation is a normal form, but not a Hesse normal form, since the normal vector $\mathbf{n} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ has length $\|\mathbf{n}\| = 3$. A normal vector of length 1 is $\frac{1}{3}\mathbf{n} = \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}$. Hence, a Hesse normal form is

$$E = \left\{\mathbf{v} \in \mathbb{R}^3 : \ \left\langle \mathbf{v} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix} \right\rangle = 0 \right\}.$$

Hesse normal form and distance

Hesse normal form (Definition)

A normal form $\{\mathbf{v} \in \mathbb{R}^3 : \langle \mathbf{v} - \mathbf{p}, \mathbf{n} \rangle = 0\}$ of a plane is called Hesse normal form if the length of $\mathbf{n}$ equals 1. ($\|n\| = 1$)

Distance point/plane

Let $\mathbf{q}$ be a point (position vector) and let E be a plane in Hesse normal form $E = \{\mathbf{v} \in \mathbb{R}^3 : \langle \mathbf{v} - \mathbf{p}, \mathbf{n} \rangle = 0\}$ then the distance between $\mathbf{q}$ and E equals

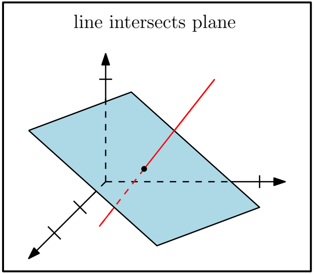
$$|\langle \mathbf{q} - \mathbf{p}, \mathbf{n} \rangle| .$$

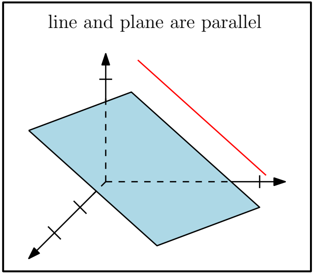
Exercise: Determine the distance between the point $\mathbf{q} := \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$ and the plane $E := \left\{ \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} : \lambda, \mu \in \mathbb{R} \right\}$.

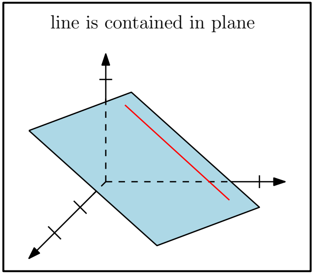
$n = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$ $\quad ||n|| = \sqrt{3^2 + 4^2 + 0^2} = 5$ $\quad \tilde{n} = \frac{1}{5} \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$ $\quad E = \left\{ v \in \mathbb{R}^3 : \langle v - \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}, \frac{1}{5}\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \rangle = 0 \right\}$

$|\langle q - p, \tilde{n} \rangle| = \left| \langle \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}, \frac{1}{5} \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \rangle \right| = \left| \langle \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}, \frac{1}{5} \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \rangle \right| = \left| \frac{20}{5} \right| = 4$

Positional relationship: line and plane







Positional relationship: line and plane

Scheme: Positional relationship between line and plane

Let a line g be given in parameter form

$$g = \{\mathbf{p} + \lambda \mathbf{a} : \ \lambda \in \mathbb{R}\}$$

and let a plane E be given in coordinate form.

Then one may put the 3 components of the general vector $\mathbf{p} + \lambda \mathbf{a}$ of the line g into the equation of the coordinate form of E. This results in an equation with variable λ for which there are three cases:

- the equation has no solution: then g and E have no common point.
- the equation has a unique solution λ: then there is an intersection point. Its position vector can be determined by substituting the solution λ into the general vector $\mathbf{p} + \lambda \mathbf{a}$.
- the equation has infinitely many solutions: g is contained in E.

Positional relationship: line and plane

Example: We check whether the following line g and the following plane E have an intersection point:

$$g = \left\{ \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \,:\, \lambda \in \mathbb{R} \right\} \quad \text{and} \quad E = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \,:\, 2x_1 - x_2 - 2x_3 = 1 \right\}.$$

Solution: Every point of g has a position vector of the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 + \lambda \\ -3 - 2\lambda \\ 0 + \lambda \end{pmatrix}.$$

Such a point lies in E if and only if it satisfies $2x_1 - x_2 - 2x_3 = 1$; hence:

$$2 \cdot (2 + \lambda) - (-3 - 2\lambda) - 2 \cdot (0 + \lambda) = 1.$$

This equation has a unique solution: $\lambda = -3$. Hence, there is an intersection point. Its position vector is

$$\begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} + (-3) \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -3 \end{pmatrix}.$$

Hence, the intersection point is $S = (-1, 3, -3)$.

Positional relationship: line and plane

Exercise: Determine the positional relationship of the following plane E and the following line g:

$$E := \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : \; 3x_1 - 4x_2 + 2x_2 = 12 \right\}$$

$$g := \left\{ \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 5 \\ 10 \end{pmatrix} : \; \lambda \in \mathbb{R} \right\}$$

Solution:

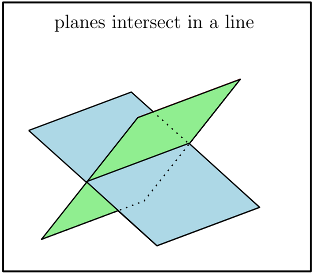
Every point of g has a position vector of the form $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 5 \\ 10 \end{pmatrix} = \begin{pmatrix} 6+4\lambda \\ 2+5\lambda \\ 1+10\lambda \end{pmatrix}$

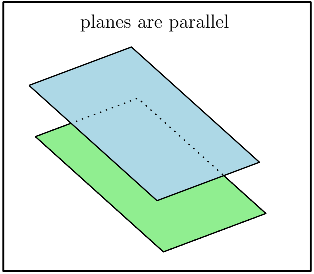
Such a point lies in E if and only if $3x_1 - 4x_2 + 2x_2 = 12$, hence

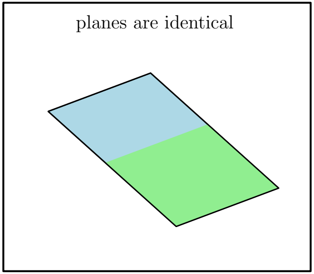
$$3 \cdot (6+4\lambda) - 4 \cdot (2+5\lambda) + 2 \cdot (1+10\lambda) = 12$$
$$\Leftrightarrow \; 18 + 12\lambda - 8 - 20\lambda + 2 + 20\lambda = 12$$
$$\Leftrightarrow \; 12 + 12\lambda = 12$$
$$\Leftrightarrow \; \lambda = 0$$

Hence, there is an intersection point, it's position vector is $\begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}$, hence the intersection point is $S = (6,2,1)$

Positional relationship: two planes







Positional relationship: two planes

Scheme: positional relationship of two planes

Assume that two planes E_1 and E_2 are given by coordinate forms. Then each common point (x_1, x_2, x_3) must satisfy both coordinate equations which leads to a LES of the form

$$\begin{aligned} a_1x_1 + a_2x_2 + a_3x_3 &= d \\ b_1x_1 + b_2x_2 + b_3x_3 &= e\,. \end{aligned}$$

There are three cases:

- the LES has no solution: then E_1 and E_2 have no common point (parallel).
- the LES has solutions, and both equations are not multiples of each other: then there is an intersection line which equals the solution set of the LES.
- the LES has solutions, and one equation is a multiple of the other equation: then the planes are identical.

Exercises later

Reminder: System of linear equations

System of linear equations

Let $m, n \in \mathbb{N}$. A system of linear equations (LES) in the variables $x_1, x_2, \ldots, x_n$ is of the form

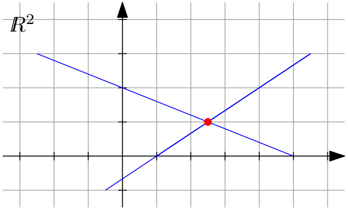
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

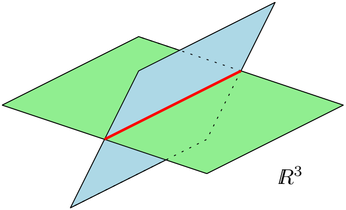
with a_{ij} and b_i being (usually real) numbers. An assignment of values for $x_1, \ldots, x_n$ such that all equations are satisfied is called a solution of this system of equations. Such a solution is written as a vector.

LES: Looking at rows

$$
\boxed{\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}}
$$

Equations describe
lines ($\mathbb{R}^2$),
planes ($\mathbb{R}^3$),
hyperplanes ($\mathbb{R}^n$)
Solution set is
their intersection





solving LES $\widehat{=}$ finding intersection of hyperplanes

LES: Looking at columns

$$\boxed{\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}}$$

$\Downarrow$ rearrange

$$\boxed{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} x_1 + \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} x_2 + \ldots + \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} x_n = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}$$

solving LES $\widehat{=}$ finding linear combination

LES: Laziness

$$\boxed{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} x_1 + \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} x_2 + \ldots + \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} x_n = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}$$

$\Downarrow$ rearrange

$$\boxed{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}$$

Real matrix

Let $m, n \in \mathbb{N}$. Then

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad \text{with every } a_{ij} \in \mathbb{R}$$

is called a real matrix. The set of all such matrices is denoted by $\mathbb{R}^{m \times n}$. m is the number of rows, n is the number of columns of the matrix $\mathbf{A}$.