Prep Course Mathematics

Solving equations

Dr. Simon Campese, Dr. Dennis Clemens, M. Sc. Yannick Mogge

TUHH

INSTITUTE OF MATHEMATICS

Equations: equivalence transformations

Equivalence transformations modify equations without altering their solutions.

Important equivalence transformations:

1. - swapping sides: $a = b \iff b = a$
2. - addition/subtraction: $a = b \iff a \pm c = b \pm c$
3. - multiplication with or division by *nonzero* constant c:

$$a = b \iff ac = bc \iff \frac{a}{c} = \frac{b}{c}$$

⚠ Taking powers or roots are **not** equivalence transformations.

1. $3x + 12 = 4$
$\iff 4 = 3x + 12$

2. $3x + 12 = 4$
$\iff 3x + 12 - 3 = 4 - 3$
$\iff 3x + 9 = 1$

3. $3x + 12 = 4$
$\iff \frac{3x}{3} + \frac{12}{3} = \frac{4}{3}$
$\iff x + 4 = \frac{4}{3}$

Solving equations / roots of functions

By subtracting one side, every equation in one unknown can equivalently be written in the form $f(x) = 0$.

Example:

$3x + 4 = 5 \quad \iff \quad 3x - 1 = 0 \quad \iff \quad f(x) = 0$ with $f(x) := 3x - 1$.

Important consequence:

The solutions of an equation are precisely the roots of the associated function.

One can often infer information of the solutions from the form of the associated function.

$$
\begin{aligned}
& 2(5x-3)+3x = 4(6-2x)+12 \\
\Leftrightarrow\ & 10x-6+3x = 24-8x+12 \\
\Leftrightarrow\ & 13x-6 = 36-8x \qquad \mid +8x \\
\Leftrightarrow\ & 21x-6 = 36 \qquad \mid -36 \\
\Leftrightarrow\ & 21x-42 = 0
\end{aligned}
$$

Linear equations

slope y-intercept

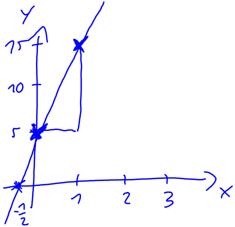
- **general form:** $ax + b = 0$ with $a \neq 0$

normal form: $x + q = 0$ with $q = \frac{b}{a}$, $a \neq 0$

- **associated function:** line $(f(x) := ax + b)$
- **solutions:** exactly one: $x = -\frac{b}{a}$ where the line intersects the x-axis

Examples.

① equation: $10x + 5 = 0$
parameters: $a = 10$, $b = 5$
solution: $x = -\frac{5}{10} = -\frac{1}{2}$



② equation: $8x - 32 = 0$
parameters: $a = 8$, $b = -32$
solution: $x = -\frac{-32}{8} = 4$

Quadratic equations

- **general form:** $ax^2 + bx + c = 0$ with $a \neq 0$
  normal form: $x^2 + px + q = 0$ with $p = \frac{b}{a}$, $q = \frac{c}{a}$, $a \neq 0$
- **associated function:** parabola
- **solutions:** none, one or two
  (intersections of the parabola with the x-axis)

There are several techniques to solve quadratic equations.

Solving quadratic equations: quadratic formula

The equation $x^2 + px + q = 0$ has roots

$$x_\pm = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}.$$

Depending on whether the radicand is negative, null or positive, there are no, one or two solutions, respectively.

Example: $x^2 + 2x - 8 = 0$

$$x_\pm = -\frac{2}{2} \pm \sqrt{\frac{2^2}{4} - (-8)}$$

$$= -1 \pm \sqrt{\frac{4}{4} + 8}$$

$$= -1 \pm \sqrt{9}$$

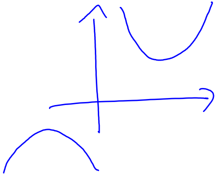
$$= -1 \pm 3$$

$$\Rightarrow x_+ = -1+3 = 2, \quad x_- = -1-3 = -4$$

Examples:

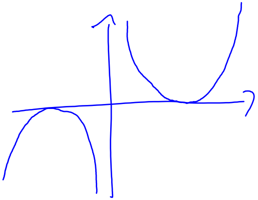
parabolas with 0 solutions

$D < 0$



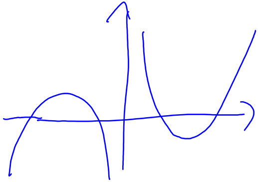
parabolas with 1 solution

$D = 0$



parabolas with 2 solutions

$D > 0$



Examples:

1. general form: $-2x^2 - 4x + 6 = 0$

   normal form. $x^2 + 2x - 3$

   parameters/radicand: $p = 2$, $q = -3$, $D = \frac{2^2}{4} - (-3) = 4 > 0$

   solutions: $x_\pm = -\frac{2}{2} \pm \sqrt{4} = -1 \pm 2$

2. geneal form! $3x^2 + 9x + 6{,}75 = 0$

   normal form: $x^2 + 3x + 2{,}25 = 0$

   parameters/radicand: $p = 3$, $q = 2{,}25$, $D = \frac{3^2}{4} - 2{,}25 = 0$

   solution: $x_\pm = -\frac{3}{2} \pm \sqrt{0} = -\frac{3}{2}$

3. general form: $x^2 - 4x + 13 = 0$

   normal form: $x^2 - 4x + 13 = 0$

   parameters/radicand: $p = -4$, $q = 13$, $D = \frac{(-4)^2}{4} - 13 = -9 < 0$

   solutions: no real solution

Solving quadratic equations: completing the square

Idea: Use binomial formula to transform $ax^2 + bx + c = 0$ into $a(x - x_0)^2 + y_0 = 0$. This yields

$$x_{1,2} = x_0 \pm \sqrt{\frac{-y_0}{a}}.$$

Depending on whether the radicand is negative, null or positive, there are no, one or two solutions, respectively.

Example: $2x^2 - 2x - 12 = 0$

$$\Leftrightarrow \quad 2(x^2 - x - 6) = 0$$

$$\Leftrightarrow \quad 2\left(x^2 - x + \left(\tfrac{1}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2 - 6\right) = 0$$

$$\Leftrightarrow \quad 2\left(\left(x - \tfrac{1}{2}\right)^2 - 6{,}25\right) = 0$$

$$\Leftrightarrow \quad 2\left(x - \tfrac{1}{2}\right)^2 - 12{,}5 = 0$$

$$\Leftrightarrow \quad \left(x - \tfrac{1}{2}\right)^2 = 6{,}25 \qquad \Big| \text{ „}\sqrt{\ }\text{"}$$

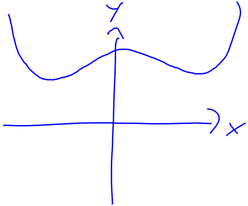
$$\Leftrightarrow \quad x - \tfrac{1}{2} = \pm\sqrt{6{,}25} \qquad \Big| + \tfrac{1}{2}$$

$$\Leftrightarrow \quad x = \tfrac{1}{2} \pm \tfrac{5}{2} \qquad x_+ = 3 \;,\; x_- = -2$$

Solving polynomial equations

How to solve equations of higher order, for example $x^5 - x - 1 = 0$?

Two techniques (only applicable in special cases):



- **Factoring:** Example: $x^3 + 2x^2 + x = 0$

  $\Leftrightarrow x \cdot (x^2 + 2x + 1) = 0$

  Zero product property:

  If $a \cdot b = 0$, then either a or b is equal to 0

- **Substitution:** Example: $x^4 - 10x^2 + 9 = 0$

  procedure: ① substitution $x^2 =: z$

  ② solving for z

  ③ re-substitution $z = x^2$

  ④ solving for x

  $z^2 - 10z + 9 = 0$

  $\Rightarrow z_\pm = -\frac{-10}{2} \pm \sqrt{\frac{(-10)^2}{4} - 9} = 5 \pm \sqrt{\frac{64}{4}}$

  $= 5 \pm 4$

  $\Rightarrow z_+ = 9, \quad z_- = 1$

  $x_{1,2} = \pm\sqrt{z_+} = \pm\sqrt{9} = \pm 3$

  $x_{3,4} = \pm\sqrt{z_-} = \pm\sqrt{1} = \pm 1$

  $x_1 = 3$
  $x_2 = -3$
  $x_3 = 1$
  $x_4 = -1$

In general:

There are (complicated) formulas for equations up to order four.
From order five onwards, no such formulas exist.

Solving radical equations

In radical equations, the variable appears under one (or more) roots, and possibly outside of roots as well.

Heuristic to solve such equations:

1. Isolate a root under which the variable appears.
2. Take squares on both sides (this might enlarge the solution set).
3. Repeat the first two steps until all roots with variable have been eliminated.
4. Solve resulting equation.
5. Check all solution candidates to eliminate false solutions.

Example: $17 + \sqrt{2x+1} - x = 0$

1) isolate the root:

$$17 + \sqrt{2x+1} - x = 0$$

$$\Leftrightarrow \quad \sqrt{2x+1} = x - 17$$

2) take square on both sides:

$$\Rightarrow \quad 2x+1 = (x-17)^2 = x^2 - 34x + 289$$

3) solve the resulting equation:

$$\Leftrightarrow \quad x^2 - 36x + 288 = 0$$

pq-formula

$$\Leftrightarrow \quad x_\pm = -\frac{-36}{2} \pm \sqrt{\frac{(-36)^2}{4} - 288}$$

$$= 18 \pm \sqrt{36}$$

$$= 18 \pm 6$$

$$x_+ = 24$$

$$x_- = 12$$

4) check all solution candidates:

$x = 24$: $17 + \sqrt{2 \cdot 24 + 1} - 24 = 0$

$\Rightarrow$ $x = 24$ is a solution

$x = 12$: $17 + \sqrt{2 \cdot 12 + 1} - 12 = 10$

$\neq 0$

$\Rightarrow$ $x = 12$ is <u>not</u> a solution

Example: $4\sqrt{x+\sqrt{x-4}} = 8$

① Isolate the root

$$4\sqrt{x+\sqrt{x-4}} = 8$$

$$\Leftrightarrow \sqrt{x+\sqrt{x-4}} = 2$$

② Take squares on both sides

$$\Rightarrow x + \sqrt{x-4} = 4$$

(1.2) Isolate the root

$$\Leftrightarrow \sqrt{x-4} = 4 - x$$

(2.2) Take squares on both sides

$$\Rightarrow x - 4 = (4-x)^2 = 16 - 8x + x^2$$

$$\Leftrightarrow x^2 - 9x + 20$$

pq-formula $\Rightarrow x_\pm = \frac{9}{2} \pm \sqrt{\frac{9^2}{4} - 20} = \frac{9}{2} \pm \sqrt{\frac{81}{4} - \frac{80}{4}}$

$$= \frac{9}{2} \pm \frac{1}{2}$$

$x_+ = \frac{10}{2} = 5 \qquad x_- = \frac{8}{2} = 4$

③ Check all solution candidates:

$x=4$: $4 \cdot \sqrt{4+\sqrt{4-4}} = 8$ ✓

$\Rightarrow x=4$ is a solution

$x=5$: $4 \cdot \sqrt{5+\sqrt{5-4}} = 4\cdot\sqrt{6} \neq 8$

$\Rightarrow x=5$ is <u>not</u> a solution

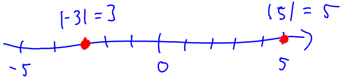
$x = 5$

$\Rightarrow x^2 = 25$

Solving equations with absolute values

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Distance from a point to zero



If absolute values appear in an equation, they can be eliminated by case-by-case analysis.

$x = -6 \quad |-6+5| = |-1| = 1$

$|-6| + 5 = 6 + 5 = 11$

Example: $|x+5| = 7$

We need to distinguish between two cases:

Case 1: $x + 5 \geq 0$

The resulting equation can be simplified under the above assumption to the equation without absolute values: $x + 5 = 7$

$\Rightarrow x = 2$

Since $x = 2$ fulfils the assumption $x + 5 > 0$, $x = 2$ is also a solution of the original equation.

Case 2: $x + 5 < 0$

The resulting equation can be calculated again, and results in $-(x+5) = 7$

$\Rightarrow -x - 5 = 7 \Rightarrow x = -12$

Since $x = -12$ fulfils the assumption $x + 5 < 0$, $x = -12$ is also a solution of the original equation.

Example: $|x^2 - 8| = 2x$

Case 1: $x^2 - 8 \geq 0$

$$x^2 - 8 = 2x$$

$$\Leftrightarrow x^2 - 2x - 8 = 0$$

$$\Leftrightarrow x_\pm = 1 \pm \sqrt{1+8} = 1 \pm 3$$

$$\Leftrightarrow x_+ = 4 \quad , \quad x_- = -2$$

Check the assumption:

x_+: $4^2 - 8 = 8 \geq 0$ ✓

x_-: $(-2)^2 - 8 = 4 - 8 = -4 < 0$ ✗

Case 2: $x^2 - 8 < 0$

$$-(x^2 - 8) = 2x$$

$$\Leftrightarrow -x^2 - 2x + 8 = 0$$

$$\Leftrightarrow x^2 + 2x - 8 = 0$$

$$\Leftrightarrow x_\pm = -1 \pm \sqrt{1+8} = -1 \pm 3$$

$$\Leftrightarrow x_+ = 2 \quad , \quad x_- = -4$$

Check the assumption:

x_+: $2^2 - 8 = -4 < 0$ ✓

x_-: $(-4)^2 - 8 = 8 > 0$ ✗

Thus, $x = 4$ and $x = 2$ are solutions of our original equation.