Prep Course Mathematics

Inequalities

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Inequalities: equivalence transformations

Inequalities are written using the comparison signs $<$, $\leqslant$, $>$, $\geqslant$.

Important equivalence transformations:

- addition/subtraction: $a < b \iff a + c < b + c$
- multiplication with/division by a positive constant:
$$a < b \iff ac < bc \iff \frac{a}{c} < \frac{b}{c}$$
where $c > 0$
- multiplication with/division by a negative constant and flipping the comparison sign:
$$a < b \iff ac > bc \iff \frac{a}{c} > \frac{b}{c}$$
where $c < 0$.
- interchanging sides and flipping the comparison sign:
$$a < b \iff b > a$$
- taking positive powers: $a < b \iff a^p < b^p$
where $a, b, p > 0$.

The above transformations are also valid using the other comparison signs.

Examples:

- $2 \geq x$

  $\Leftrightarrow x \leq 2$

- $x+3 > 5 \quad | -3$

  $\Leftrightarrow x+3-3 > 5-3$

  $\Leftrightarrow x > 2$

- $2x < x \quad | -x$

  $\Leftrightarrow 2x - x < 0$

  $\Leftrightarrow x < 0$

- $x^2 - 7 \leq 4 \quad | +7$

  $\Leftrightarrow x^2 \leq 11 \quad | -4$

  $\Leftrightarrow x^2 - 4 \leq 7$

- $\frac{x}{3} \geq -6 \quad | \cdot 3$

  $\Leftrightarrow x \geq -18$

- $\frac{x^2}{10} + \frac{x}{5} > 7 \quad | \cdot 10$

  $\Leftrightarrow x^2 + 2x > 70$

- $-\frac{x^2}{6} + \frac{x}{3} - 1 > 0 \quad | \cdot (-6)$

  $\Leftrightarrow x^2 - 2x + 6 < 0$

- $x \geq 9 \Leftrightarrow x^2 \geq 81 \Leftrightarrow x^{\frac{1}{2}} = \sqrt{x} \geq 9^{\frac{1}{2}} = \sqrt{9} = 3$

- $x \leq 36 \Leftrightarrow \sqrt{x} \leq 6$

- $x < 7 \Leftrightarrow x^2 < 49$

Solving inequalities

Inequalities typically have *infinitely many solutions*.

Those are typically calculated using one of the following **two approaches:**

- Use equivalence transformations to isolate the variable.
  Example:

  $$2x + 3 > 7 \quad \Longleftrightarrow \quad 2x > 4 \quad \Longleftrightarrow \quad x > 2$$

- Solve the associated equation and then check values in between the solutions.
  Example: $x^2 + 2x - 1 < 2$

Linear inequalities

- inequalities of the form $ax+b \begin{Bmatrix} < \\ \leq \\ > \\ \geq \end{Bmatrix} cx+d$ with real constants a, b, c, d

Solving numerically

↳ perform equivalence transformations

Examples:

① $2x-3>5$
$\Leftrightarrow 2x>8$
$\Leftrightarrow x>4$

② $-8x+6 \geq 0$
$\Leftrightarrow -8x \geq -6$
$\Leftrightarrow x \leq \frac{3}{4}$

③ $2x+5 \leq 2x-3$
$\Leftrightarrow 5 \leq -3$ ↯

This inequality is false for all values of x, thus there are no valid solutions for x.

Exercise:

Transform the inequality so that the left side is x and on the right side there are no multiples of x

$$5x+12 \leq -5(x+1)+3$$

$$\Leftrightarrow 5x+12 \leq -5x-5+3 = -5x-2 \quad | +5x$$

$$\Leftrightarrow 10x+12 \leq -2 \quad | -12$$

$$10x \leq -14 \quad | :10$$

$$x \leq -\frac{14}{10} = -\frac{7}{5} = -1.4$$

Quadratic inequalities

inequalities of the form $ax^2+bx+c \begin{Bmatrix} < \\ \leq \\ > \\ \geq \end{Bmatrix} dx^2+fx+g$ with real constants a, b, c, d, f, g

Every inequality can be transformed into an equivalent form with zero on the right hand side.

Example: $x^2+1 < 3x^2-9x+8 \quad | -3x^2+9x-8$

$\Leftrightarrow -2x^2+9x-7 < 0$

Procedure: ① Find the "$=0$" points

In between the "$=0$" points, the intervals are either

- greater than zero

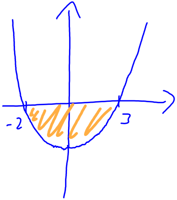
or

- smaller than zero

② Pick a test value to find out, which one holds (>0 or <0)

Examples: (Quadratic term with two roots)

$$x^2 - x - 6 \overset{>}{<} 0$$
$$\shortparallel$$
$$(x+2)(x-3) \overset{>}{<} 0$$



① It is equal to zero at $x = -2$ and $x = 3$

② At $x = 0$: $0^2 - 0 - 6 = -6 < 0$

So the inequality "<0" is true between -2 and 3

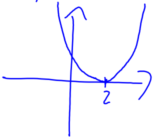
The inequality is fulfiled if
$-2 < x < 3$

$x < -2$ or $x > 3$

Examples: (Quadratic term with only one root)

$$x^2 > 4x - 4$$

$\Leftrightarrow$ $x^2 - 4x + 4 > 0$



① $x^2 - 4x + 4 = 0$

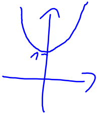
$\Leftrightarrow$ $x_\pm = 2 \pm \sqrt{2^2 - 4} = 2$

② At $x = 3$: $3^2 - 4 \cdot 3 + 4 = 9 - 12 + 4 = 1 > 0$

Thus the inequality ">0" is true for $x \neq 2$

Question: what if it doesn't go through zero at all?

$$x^2 + 1 \overset{<}{>} 0$$



② Check $x = 0$

$0^2 + 1 = 1 > 0$

The inequality is true for all values of x.

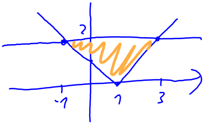
no

Absolute values in inequalities

(absolute value of linear terms, i.e. terms of the form $|cx+b|$ with real constants b,c

Example: $|x-1| < 2$

graphic solution:



The inequality is true for $-1 < x < 3$

Numerical solution:

↳ Case distinction

Example: $|x-2| < 4$

$$|x-2| = \begin{cases} x-2 & \text{if } x-2 \geq 0 \iff x \geq 2 \\ -(x-2) = 2-x & \text{if } x-2 < 0 \iff x < 2 \end{cases}$$

Case 1: $x \geq 2$

$x - 2 < 4$

$\iff x < 6$

Thus the inequality holds if $x \geq 2$ and $x < 6$.

Case 2: $x < 2$

$2 - x < 4$

$\iff -2 < x$

Thus the inequality holds if $x > -2$ and $x < 2$

$\Rightarrow$ The inequality $|x-2| < 4$ is true for all x that satisfy both $x > -2$ and $x < 6$

$-2 < x < 6$

Example: $|3x-7| \geq 1$

$$|3x-7| = \begin{cases} 3x-7 & \text{if } 3x-7 \geq 0 \Leftrightarrow x \geq \frac{7}{3} \\ 7-3x & \text{if } 3x-7 < 0 \Leftrightarrow x < \frac{7}{3} \end{cases}$$

Case 1: $x \geq \frac{7}{3}$

$$3x-7 \geq 1$$
$$\Leftrightarrow 3x \geq 8$$
$$\Leftrightarrow x \geq \frac{8}{3}$$

Because $\frac{8}{3} > \frac{7}{3}$, the strongest restriction is $x \geq \frac{8}{3}$

Hence, all $x \geq \frac{8}{3}$ fulfil the inequality

Case 2: $x < \frac{7}{3}$

$$7-3x \geq 1$$
$$\Leftrightarrow -3x \geq -6$$
$$\Leftrightarrow x \leq 2$$

Because $2 < \frac{7}{3}$, the strongest restriction is $x \leq 2$

Hence all $x \leq 2$ fulfil the inequality

Thus, the inequality $|3x-7| \geq 1$ is true for all $x \geq \frac{8}{3}$ or $x \leq 2$.

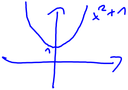
Inequalities with the variable in the denominator of a fraction

It is often useful to multiply the inequality by the denominator

- If the denominator has the same sign for all x, then we get an equivalent inequality.
- If the denominator has a different sign for different values of x, then a case distinction is necessary.

(1) Example

$$1 \leq \frac{2}{x^2+1}$$



The denominator is positive for all x.

$$1 \leq \frac{2}{x^2+1} \quad | \cdot (x^2+1)$$

$x^2+1 > 0$
$\Leftrightarrow x^2 > -1$ ✓
since $x^2 \geq 0$ for all x

$\Leftrightarrow x^2+1 \leq 2$
$\Leftrightarrow x^2 \leq 1$
$\Leftrightarrow x^2-1 \leq 0$

The quadratic equation $x^2-1=0$ has the solutions $x_1=-1$ and $x_2=1$

Check $x=0$: $0^2-1=-1 \leq 0$ ✓

Thus the inequality $1 \leq \frac{2}{x^2+1}$ is true for all x that satisfy $x \geq -1$ and $x \leq 1$.

(2) $$\frac{1}{1-x} > 3$$

The root of the denominator ($x=1$) needs to be excluded.

- For $x<1$ the denominator is positive.
- For $x>1$ the denominator is negative

Case 1: $x<1$

$\frac{1}{1-x} > 3$
$\Leftrightarrow 1 > 3(1-x) = 3-3x$
$\Leftrightarrow -2 > -3x$
$\Leftrightarrow \frac{2}{3} < x$

Case 2: $x>1$

$\frac{1}{1-x} > 3$
$\Leftrightarrow 1 < 3(1-x) = 3-3x$
$\Leftrightarrow -2 < -3x$
$\Leftrightarrow \frac{2}{3} > x$

The inequality of the first case is fulfilled for $\frac{2}{3} < x < 1$

In the second inequality, the conditions $x > 1$ and $x < \frac{2}{3}$ contradict each other.

Consequently, $\frac{1}{1-x} > 3$ is fulfilled for all $\frac{2}{3} < x < 1$