Prep Course Mathematics

Trigonometry

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What is an angle?

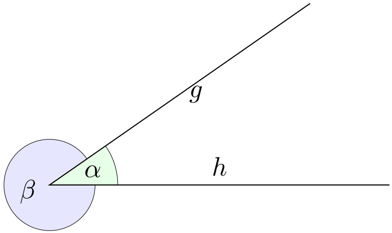


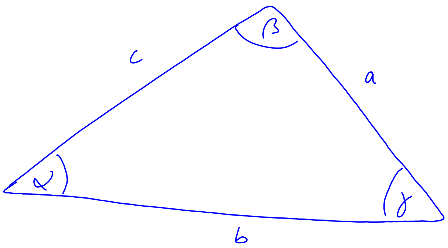
Figure: Angle between two half-lines (also called rays)

angle = measurement of rotation

angle measurements:

- degree
- radian

The sum of the interior angles in a triangle is always 180° (radian π).



Let $\alpha = 46°$ and $\beta = 30°$

What is γ? $\alpha + \beta + \gamma = 180°$

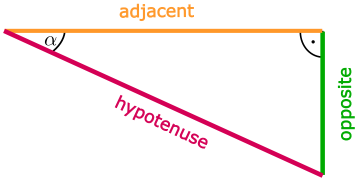
$$\Leftrightarrow \gamma = 180° - \alpha - \beta = 180° - 46° - 30°$$

$$= 104°$$

Names of angles

	**full angle**	half lines overlap	360°
	**straight angle**	half lines form a straight line	180°
	**right angle**	half the size of straight angle	90°
	**acute angle**	smaller than right angle	
	**obtuse angle**	bigger than right angle, but smaller than straight angle	
	**reflex angle**	bigger than straight angle	

Right-angled triangle



The ratio of these sides to each other defines the **trigonometric functions**.

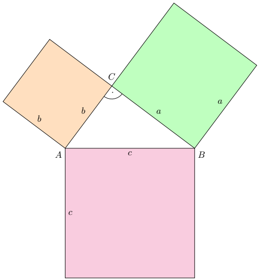
$$\sin(\alpha) := \frac{\text{opposite}}{\text{hypotenuse}} \qquad \tan(\alpha) := \frac{\text{opposite}}{\text{adjacent}}$$

$$\cos(\alpha) := \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \cot(\alpha) := \frac{\text{adjacent}}{\text{opposite}}$$

$\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)}$

Pythagorean theorem

$$a^2 + b^2 = c^2$$

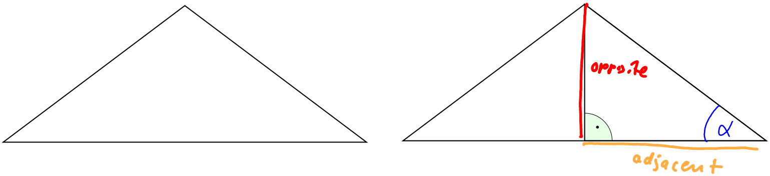


General triangle

Based on our definition, we can calculate,
for example, the sine of an angle in a **general triangle**

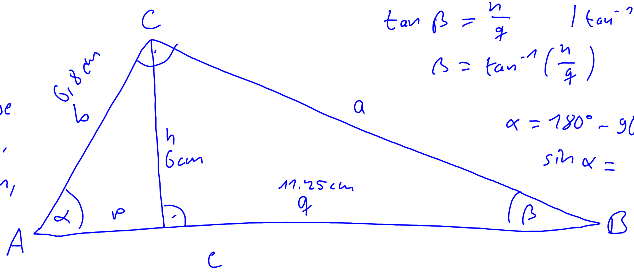
$$\Downarrow$$

Drawing **right-angled auxiliary triangles**



Exercise

In the right angled triangle ABC, whose right angle is at C, the altitude $h = 6\,cm$, the side $b = 6{,}8\,cm$, the part q of the hypothenuse is $q = 11.25\,cm$ are given.



$\tan\beta = \frac{h}{q} \quad | \tan^{-1}$

$\beta = \tan^{-1}\left(\frac{h}{q}\right)$

$\alpha = 180° - 90° - \beta$

$\sin\alpha = \frac{h}{b}$

$b = \frac{h}{\sin\alpha}$

Calculate the sides a and p and the hypothenuse c

Use pythagoras: $a^2 = h^2 + q^2$

$b^2 = p^2 + h^2$

$p^2 = b^2 - h^2 = (6{,}8\,cm)^2 - (6\,cm)^2$

$= 10{,}24\,cm^2$

$\Rightarrow p = \sqrt{10{,}24\,cm^2} = 3{,}2\,cm$

$\Rightarrow c = p + q = 11{,}25\,cm + 3{,}2\,cm$

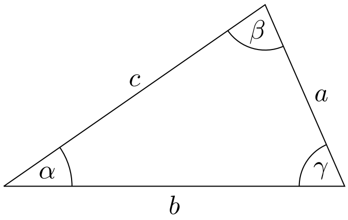
$= 14{,}45\,cm$

$a^2 + b^2 = c^2$

$a = \sqrt{c^2 - b^2} = \sqrt{(14{,}45\,cm)^2 - (6{,}8\,cm)^2}$

$= 12{,}75\,cm$

General triangle



Further useful results for calculating side lengths or angles in general triangles:

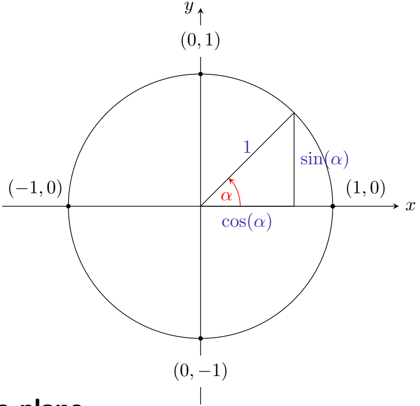
Sine rule:

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$$

Cosine rule:

$$a^2 + b^2 - 2ab\cos(\gamma) = c^2$$

Unit circle: values for sine and cosine



Unit circle in the plane

= circle with radius 1 around the points $(0,0)$

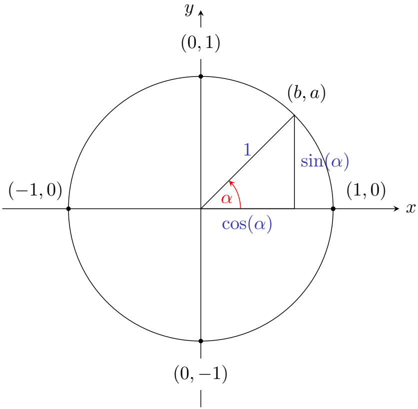
drawn triangle has a right angle

$\Downarrow$

Sine and cosine values can be read off:

- sine value of α on the y-axis
- cosine value of α on the x-axis

Angle measurement: radian



Starting point: unit circle in the plane

length of the complete arc
$=$ circumference of a circle with radius 1
$= 2\pi$

angle $\alpha =$ length of the arc between $(1,0)$ and (b,a)

Conversion from degree to radian

$$\frac{\text{degrees}}{360^\circ} = \frac{\text{radians}}{2\pi}$$

Let α be an angle which has as degree the form x°.
Then radian can be calculated as follows:

$$\alpha = \frac{x}{180} \cdot \pi.$$

degrees	0°	30°	45°	60°	90°	180°	270°	360°
radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

Exercises

Convert the following angles from degrees to radians:

- $30^\circ = \frac{30^\circ}{180^\circ}\pi = \frac{\pi}{6} \approx 0.52$
- $50^\circ = \frac{50^\circ}{180^\circ}\pi = \frac{5\pi}{18} \approx 0.82$
- $122^\circ = \frac{61\pi}{90} \approx 2.13$
- $315^\circ = \frac{7\pi}{4} \approx 5.50$

Convert the following radians from degrees to angles:

- $\frac{3\pi}{4} = \frac{3\cancel{\pi}\cdot 180^\circ}{4\cancel{\pi}} = 135^\circ$
- $0{,}36\pi = \frac{0{,}36\cancel{\pi}\cdot 180^\circ}{\cancel{\pi}} = 64.8^\circ$

Trigonometric functions

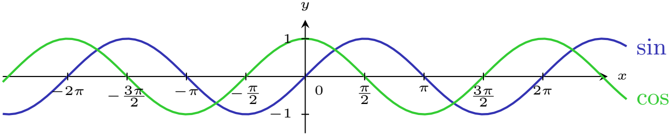
Sine function

$$\sin\colon \mathbb{R} \to \mathbb{R}$$

Properties:

$\sin(x) = -\sin(-x)$

- antisymmetric, i.e. $\sin(-x) = -\sin(x)$
- zeros: $\{k \cdot \pi : k \in \mathbb{Z}\}$
- periodic with period 2π, i.e. $\sin(x) = \sin(x + 2\pi)$



Cosine function

$$\cos\colon \mathbb{R} \to \mathbb{R}$$

Properties:

- symmetric, i.e. $\cos(x) = \cos(-x)$
- zeros: $\{\frac{\pi}{2} + k \cdot \pi : k \in \mathbb{Z}\}$
- periodic with period 2π, i.e. $\cos(x) = \cos(x + 2\pi)$

Trigonometric functions

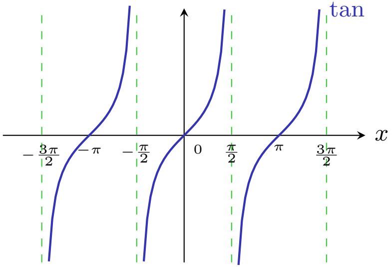
Tangent function

$$\tan\colon \{x \in \mathbb{R} : \cos(x) \neq 0\} \to \mathbb{R}$$

$\tan(x) = \frac{\sin(x)}{\cos(x)}$

Properties:

- antisymmetric
- periodic with period π,
  i.e. $\tan(x) = \tan(x + \pi)$
- zeros: $\{k \cdot \pi : k \in \mathbb{Z}\}$



Relations between trigonometric functions

- $\cos(x) = \sin\left(x + \frac{\pi}{2}\right)$
- $\sin(x) = \cos\left(x - \frac{\pi}{2}\right)$
- $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
- $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
- $\sin^2(x) + \cos^2(x) = 1$