

# Prep Course Mathematics

## Logic

Dr. Simon Campese, Dr. Dennis Clemens, M. Sc. Yannick Mogge



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# Logical statement

## Definition (Logical statement)

A **logical statement** (or proposition) is a statement, which means a meaningful declarative sentence, that is either *true* or *false*.

**Example:** Which of these are statements?

- ▶ Each natural number is also an integer.
- ▶ Please open the door.
- ▶  $3 > 7$
- ▶ How are you?
- ▶  $1 + 2$


# Logical operations

Let  $A$  and  $B$  be two logical statements.

## Conjunction

The statement „ $A$  **and**  $B$  “ ...

- ▶ is true, if both statements  $A$  and  $B$  are true.
- ▶ is false, if at least one of the statements  $A$  and  $B$  is false.

  $A \wedge B$

$A$	$B$	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

**Example:**  $A :=$  The food is sweet.     $B :=$  The food is sour.

$A \wedge B =$

## Disjunction

The statement “ $A$  **or**  $B$ ” ...

- ▶ is true, if at least one of the statements  $A$  and  $B$  is true.
- ▶ is false, if both statements  $A$  and  $B$  are false.

  $A \vee B$

$A$	$B$	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

**Example:**  $A :=$  Today the sun is shining.     $B :=$  Today it is raining.


What is the weather like, if we have  $A \vee B$ ?

# Logical operations

## Implication (If ..., then ...)

The statement “ $A$  **implies**  $B$ ” ...

- ▶ is false, if  $A$  is true and  $B$  is false.
- ▶ is true, if  $A$  is false or  $B$  is true.

  $A \Rightarrow B$

$A$	$B$	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

### Example:

- ▶ It's raining.  $\Rightarrow$  The road is wet.
- ▶ The mobile phone rings.  $\Rightarrow$  The battery is not empty.

### Other ways describing “ $A \Rightarrow B$ ” in words

- ▶ From  $A$  follows  $B$ .
- ▶  $A$  is *sufficient* for  $B$ .
- ▶ Without  $B$ ,  $A$  cannot occur.
- ▶  $B$  is *necessary* for  $A$ .
- ▶  $A$  implies  $B$ . (implies = results in)

## Example: Implication

$A$	$B$	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

A grandmother makes the following promise to her grandson:

„If you pass the math test,  
I'll give you a bicycle.“

When is the grandma lying?

The grandma's promise is of the form  $A \Rightarrow B$  with

- ▶ Statement A „The grandson passes the math test“,
- ▶ Statement B: „The grandma buys a bicycle for her grandson“.

The grandma is lying if A is fulfilled and B is not (line 2).

If the grandma buys a bicycle, she is not lying (line 1).

If the grandson does not pass the test, then the grandma cannot break her promise (line 3, 4).

# Logical operations

## Equivalence (...if and only if ...)

The statement “ $A$  is **equivalent** to  $B$ ” ...

- ▶ is true if either both statements  $A$  and  $B$  are true or both statements  $A$  and  $B$  are false.
- ▶ is false if one of the statements  $A$  and  $B$  is true when the other is false.

  $A \Leftrightarrow B$

$A$	$B$	$A \Leftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

**Example:** The exam is passed if and only if at least 50% of the points have been achieved.

## Other ways describing “ $A \Leftrightarrow B$ ” in words

- ▶ From  $A$  follows  $B$  and from  $B$  follows  $A$ .
- ▶  $A$  holds if and only if  $B$  holds.
- ▶  $A$  is *sufficient* and *necessary* for  $B$ .

# Negation

## Negation

The negation of  $A$  is the statement, which ...

- ▶ is true if  $A$  is false.
- ▶ is false if  $A$  is true.

$A$	$\neg A$
T	F
F	T



$\neg A$

### Observation:

$B$  is the negation of  $A$  if and only if both statements  $A$  and  $B$  are never true at the same time and never false at the same time.

### Example:

What is the negation of the following logical statement?

$R :=$  „In this room there is a red chair.“

- ▶  $A :=$  „There are no chairs in this room.“
- ▶  $B :=$  „In this room there is a chair that is not red.“
- ▶  $C :=$  „All chairs in this room are Green.“
- ▶  $D :=$  „All chairs in this room are not red.“



# Negation of statement with AND/OR

Let  $A$  and  $B$  be two statements.

- ▶ The negation of the statement „ $A$  **and**  $B$ “ is „**not**  $A$  **or** **not**  $B$ “.
- ▶ The negation of the statement „ $A$  **or**  $B$ “ is „**not**  $A$  **and** **not**  $B$ “.

**Reason:**

$A$	$B$	$A \wedge B$ $A$ and $B$	$\neg A$ not $A$	$\neg B$ not $B$	$(\neg A) \vee (\neg B)$ (not $A$ ) or (not $B$ )
T	T	T	F	F	F
T	F	F	F	T	T
F	T	F	T	F	T
F	F	F	T	T	T

# Predicates and quantifiers

## Definition (Predicate)


A **predicate** is a statement that may be true or false depending on the values of its variables.


**Example:** Let  $A(x, y)$  be the predicate „The double of  $x$  is less than  $y$ .“

►  $A(2, 5)$  is true

►  $A(3, 5)$  is false

Quantifiers:  $\forall$  - for all     $\exists$  - it exists

	Meaning
$\forall x : A(x)$	„For all $x$ $A(x)$ is true.“
$\exists x : A(x)$	„It exists at least one $x$ for which $A(x)$ is true.“
$\exists! x : A(x)$	„It exists exactly one $x$ for which $A(x)$ is true.“

 If we say „It exists ...“, then we mean: „It exists at least one ...“.

**Example:**

„Every natural number is greater than 0.“



# Negation of the quantifiers

A statement with quantifiers and a predicate  $A(x, y, z, \dots)$  is given.

Then we get its **negation** by

- ▶ replacing  $\forall$ -quantifier by  $\exists$ -quantifier,
- ▶ replacing  $\exists$ -quantifier by  $\forall$ -quantifier and
- ▶ negating  $A(x, y, z, \dots)$ .



## Rule of thump

$$\neg \forall = \exists \neg \quad \text{and} \quad \neg \exists = \forall \neg$$

### Example:

„It exists a natural number  $n$  so that every natural number  $m$  is at least as large as  $n$ .“



Negation:

