Prep Course Mathematics

Differentiation

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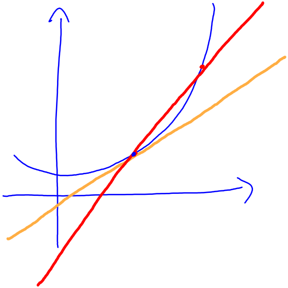
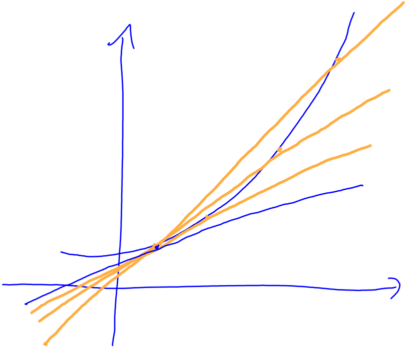
Tangents and secants



The rates of change can be represented by tangents of the curve.

A line is only uniquely defined by two points.

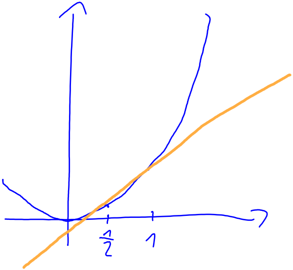
Question: How do we determine a tangent that touches the curve at only one point?



Answer: By a sequence of secants

Secant = line that intersects the curve at two points

Tangents and secants

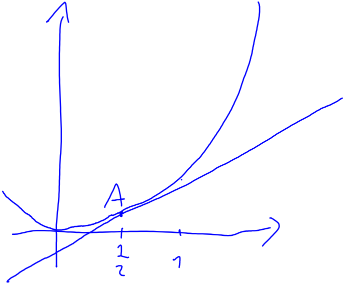


Example: $f(x) := \frac{1}{2}x^2$

Secants through the points

$A := \left(\frac{1}{2}, f\left(\frac{1}{2}\right)\right)$ and $\left(\frac{1}{2}+h, f\left(\frac{1}{2}+h\right)\right)$

$h = \frac{1}{2}$

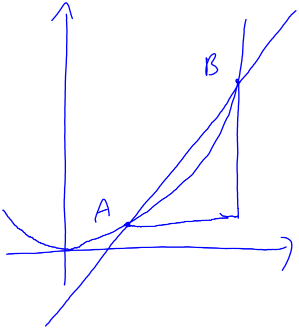


Tangent at the point $A = \left(\frac{1}{2}, f\left(\frac{1}{2}\right)\right)$

Obserservation:

The sequence of secants approaches for decreasing h the tangent of the graph for f at point A.

Slope of the secant = difference quotient



secant through the points $A = (x, f(x))$

$B = (z, f(z))$

difference quotient $\dfrac{f(z) - f(x)}{z - x}$

A function $f: D \to W$ is called <u>differentiable</u> at $x \in D$,

if the limit $\lim\limits_{z \to x} \dfrac{f(z) - f(x)}{z - x}$ exists $\left(\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} \right)$

The limit is called the <u>derivative at x</u>.

Notation: $f'(x)$, $\frac{d}{dx} f(x)$, ...

A function is called <u>differentiable</u> if it is differentiable at all points of the domain.

Derivatives of elementary functions and rules for derivatives

$f(x)$	$f'(x)$
$c \quad (c \in \mathbb{R})$	0
$x^\alpha \quad (\alpha \neq 0)$	$\alpha \cdot x^{\alpha-1}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\frac{1}{\cos^2(x)}$
$\cot(x)$	$-\frac{1}{\sin^2(x)}$
e^x	e^x
$a^x \quad (a > 0)$	$\ln(a) \cdot a^x$
$\ln(x)$	$\frac{1}{x}$
$\log_a(x) \quad (a > 0)$	$\frac{1}{\ln(a) \cdot x}$

Constant factor rule: for $c \in \mathbb{R}$

$$(c \cdot f)'(x) = c \cdot f'(x)$$

$\sqrt{x} = x^{\frac{1}{2}} \rightarrow \frac{1}{2} x^{-\frac{1}{2}}$

Sum rule:

$$(f \pm g)'(x) = f'(x) \pm g'(x)$$

$= \frac{1}{2} \cdot \frac{1}{\sqrt{x}}$

Product rule:

$$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Quotient rule:

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Chain rule:

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

Examples

$x^1 \longrightarrow 1 \cdot x^0 = 1 \cdot 1 = 1$

(1) $f(x) := x^4 - \frac{3}{2} \cdot x^2 + 5x - 4$

$f'(x) = 4x^3 - \frac{3}{2} \cdot 2 \cdot x + 5 \cdot 1 = 4x^3 - 3x + 5$

(2) $f(x) := \frac{4x - 14}{\sqrt{x}} = \frac{4x - 14}{x^{\frac{1}{2}}} = \frac{4x}{x^{\frac{1}{2}}} - \frac{14}{x^{\frac{1}{2}}} = 4x^{\frac{1}{2}} - 14x^{-\frac{1}{2}}$

$f'(x) = 2x^{-\frac{1}{2}} + 7x^{-\frac{3}{2}}$

Using Quotient rule. $\frac{4 \cdot x^{\frac{1}{2}} - (4x - 14) \cdot \frac{1}{2} \cdot x^{-\frac{1}{2}}}{(x^{\frac{1}{2}})^2} = \frac{4x^{\frac{1}{2}} - 2x^{\frac{1}{2}} + 7x^{-\frac{1}{2}}}{x}$

$= 2x^{-\frac{1}{2}} + 7x^{-\frac{3}{2}}$

(3) $f(x) := (3x^2 + 5x + 2)^7$

$f'(x) = 7 \cdot (3x^2 + 5x + 2)^6 \cdot (6x + 5)$

(4) $f(x) := \sin(3x)$

$f'(x) = \cos(3x) \cdot 3 = 3\cos(3x)$

(5) $f(x) := e^{x^3 - 3x}$

$f'(x) = e^{x^3 - 3x} \cdot (3x^2 - 3) = 3 \cdot (x^2 - 1) e^{x^3 - 3x}$

⑥ $f(x) := \frac{3}{\sqrt{x^3-5}} = 3(x^3-5)^{-\frac{1}{2}}$

$f'(x) = -\frac{1}{2} \cdot 3 \cdot (x^3-5)^{-\frac{3}{2}} \cdot (3x^2) = -\frac{9}{2} x^2 (x^3-5)^{-\frac{3}{2}}$

⑦ $f(x) := (\cos(\ln(2x)))^3$

$u(x) = \cos(\ln(2x))$

$u'(x) = -\sin(\ln(2x)) \cdot \frac{1}{x}$

$v(x) = \ln(2x)$

$v'(x) = \frac{1}{2x} \cdot 2 = \frac{1}{x}$

$f'(x) = 3(\cos(\ln(2x)))^2 \cdot \left(-\sin(\ln(2x)) \cdot \frac{1}{x}\right)$

$= -\frac{3}{x} (\cos(\ln(2x)))^2 \cdot (\sin(\ln(2x)))$

⑧ $f(x) := x^4(3x^2-5)^5$

$u(x) = x^4$ $\quad v(x) = (3x^2-5)^5$

$u'(x) = 4x^3$ $\quad v'(x)\ 5(3x^2-5)^4\ 6x = 30x(3x^2-5)^4$

$f'(x) = u'(x) \cdot v(x) + u(x) \cdot v'(x)$

$= 4x^3 \cdot (3x^2-5)^5 + 30x^5(3x^2-5)^4$

⑨ $f(x) := \frac{x^2-4x+12}{(x-3)^2}$

$u(x) = x^2-4x+12$ $\quad v(x) = (x-3)^2$

$u'(x) = 2x-4$ $\quad v'(x) = 2(x-3)$

$f'(x) = \frac{u'(x) \cdot v(x) - u(x) \cdot v'(x)}{(v(x))^2} = \frac{(2x-4) \cdot (x-3)^2 - (x^2-4x+12) \cdot 2(x-3)}{(x-3)^4}$

$= \frac{(2x-4)(x-3) - 2(x^2-4x+12)}{(x-3)^3}$

Higher derivatives

① $f(x) := 3x^4 - 2x^3 - 2x^2 + 5x + 19$

$f'(x) = 12x^3 - 6x^2 - 4x + 5$

$f''(x) = 36x^2 - 12x - 4$

$f'''(x) = 72x - 12$

② $f(x) := \sin(x^2 + 3x)$

$u(x) = \cos(x^2+3x)$ $\quad v(x) = 2x+3$

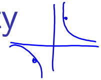
$u'(x) = -\sin(x^2+3x)\cdot(2x+3)$ $\quad v'(x) = 2$

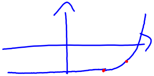
$f'(x) = \cos(x^2+3x)\cdot(2x+3)$

$f''(x) = u'(x)\cdot v(x) + u(x)\,v'(x) = -\sin(x^2+3x)(2x+3)\cdot(2x+3) + \cos(x^2+3x)\cdot 2$

$= 2\cos(x^2+3x) - \sin(x^2+3x)\cdot(2x+3)^2$

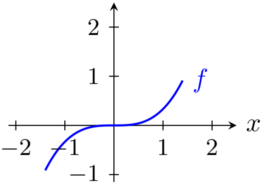
Differentiability and monotonicity

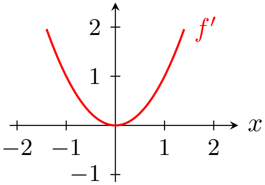




For $f\colon D \to \mathbb{R}$ differentiable:

- $f'(x) \geqslant 0$ for all $x \in D \iff f$ monotonically increasing
- $f'(x) > 0$ for all $x \in D \implies f$ strictly monotonically increasing
- $f'(x) \leqslant 0$ for all $x \in D \iff f$ monotonically decreasing
- $f'(x) < 0$ for all $x \in D \implies f$ strictly monotonically decreasing





⚠ Note that $f'(x) > 0$ resp. $f'(x) < 0$ is only sufficient, but not a necessary condition for strict monotonicity.

Example: $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) := x^3$, str. mon. increasing, but $f'(0) = 0$

Examples

(1) Let f be defined by $f(x) := c$, where $c \in \mathbb{R}$

$f'(x) = 0$ $\quad (0 \leq 0$ and $0 \geq 0)$

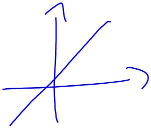
Hence, f is monotonically increasing and decreasing

(2) Let f be defined by $f(x) := e^{-x}$

$f'(x) = -e^{-x} < 0$

Hence, f is strictly monotonically decreasing

Second derivative and convexity/concavity

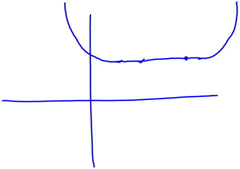


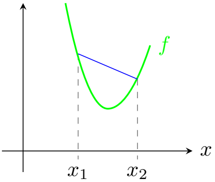
For D interval, $f\colon D \to \mathbb{R}$:

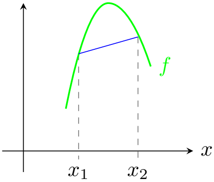
- f convex, if for all $x_1, x_2 \in D$ and $\lambda \in (0,1)$:

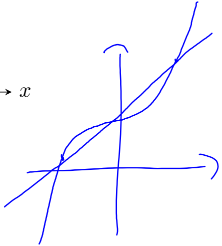
$$f(\lambda x_1 + (1-\lambda)x_2) \leqslant \lambda f(x_1) + (1-\lambda) f(x_2).$$

- f strictly convex if "$<$" instead of "$\leqslant$"
- f (strictly) concave, if $-f$ (strictly) convex, i.e. "$\geqslant$" ("$>$") instead of "$\leqslant$" ("$<$")









For f two times differentiable:

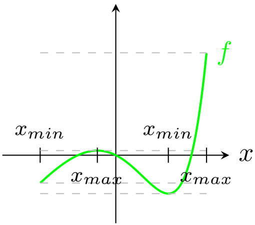
- $f''(x) \geqslant 0$ for all $x \in D \iff f$ convex.
- $f''(x) > 0$ for all $x \in D \implies f$ strictly convex.
- $f''(x) \leqslant 0$ for all $x \in D \iff f$ concave.
- $f''(x) < 0$ for all $x \in D \implies f$ strictly concave.

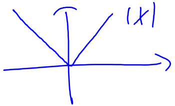
Local extrema

For $D \subset \mathbb{R}$ and function $f \colon D \to \mathbb{R}$:

$x_0 \in D$ is called

- local maximum if there exists a neighbourhood U around x_0 such that $f(x) \leqslant f(x_0)$ for $x \in U$
- local minimum if there exists a neighbourhood U around x_0 such that $f(x) \geqslant f(x_0)$ for $x \in U$
- local extremum if it is a local maximum or a local minimum.
- global maximum if $f(x) \leqslant f(x_0)$ for all $x \in D$.
- global minimum if $f(x) \geqslant f(x_0)$ for all $x \in D$.





Neighbourhood around U = (arbitrarily small) open subinterval of D that contains x_0.

Theorem

For $f \colon D \to \mathbb{R}$ differentiable, $x_0 \in D$ interior point: If f has a local extremum at x_0, then $f'(x_0) = 0$.

⚠ For local extrema at boundary points x_0 we can have $f'(x_0) \neq 0$.

$x_0 \in D$ stationary point if $f'(x_0) = 0$.

Criteria for extrema

For $f\colon (a,b) \to \mathbb{R}$ differentiable, $x_0 \in (a,b)$ stationary point:

...using the first derivative:

Theorem

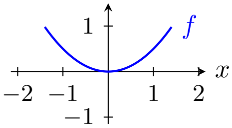
- *If $f'(x) \geqslant 0$ for $x < x_0$ and $f'(x) \leqslant 0$ for $x > x_0$, then f has a local maximum in x_0.*
- *If $f'(x) \leqslant 0$ for $x < x_0$ and $f'(x) \geqslant 0$ for $x > x_0$, then f has a local minimum in x_0.*

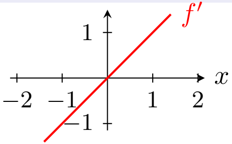
...using the second derivative:

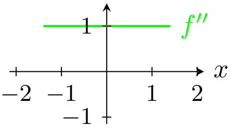
Theorem

For f two times differentiable:

- *If $f''(x_0) < 0$, then f has a local maximum in x_0.*
- *If $f''(x_0) > 0$, then f has a local minimum in x_0.*







Example)

1) where are the extrema of the function $f(x) := x^2 e^{-x}$?

$$f'(x) = 2x \cdot e^{-x} - x^2 \cdot e^{-x} = (2x - x^2)e^{-x} = 0$$

Hence, candidates for extrema: $x_1 = 0$, $x_2 = 2$

$$f''(x) = (2-2x)e^{-x} - (2x - x^2)e^{-x} = (2 - 4x + x^2)e^{-x}$$

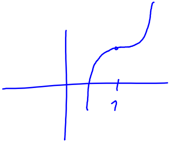
$f''(0) = 2 \cdot e^{-0} = 2 > 0 \Rightarrow$ minimum at $x = 0$

$f''(2) = -2e^{-2} < 0 \Rightarrow$ maximum at $x = 2$

2) where are the extrema of the function $f(x) := x^3 - 3x^2 + 3x + 1$

$$f'(x) = 3x^2 - 6x + 3 = 3(x-1)^2 = 0$$

Hence, candidates for extrema: $x = 1$



$$f''(x) = 6x - 6$$

$$f''(1) = 0$$

Is there a change of signs in the first derivative?

$f'(x) = 3(x-1)^2 > 0$ for all $x \neq 1$

Hence, there is no extremum at $x = 1$

Inflection points

For $f\colon (a,b) \to \mathbb{R}$, $x_0 \in (a,b)$:
f has inflection point at x_0, if the second derivative changes its sign.

Theorem (Criteria for inflection points)

- *If f is two times differentiable and has an inflection point at x_0, then $f''(x_0) = 0$.*
- *If f is three times differentiable, $f''(x_0) = 0$ and $f'''(x_0) \neq 0$, then f has an inflection point at x_0.*

Examples

(1) where are the inflection points of the function $f(x) := x^3 - 3x^2 + 3x + 1$

$f'(x) = 3x^2 - 6x + 3 = 3(x-1)^2$

$f''(x) = 6x - 6 = 0$

candidates for inflection points: $x = 1$

$f'''(x) = 6$

$f'''(1) = 6 > 0 \Rightarrow$ There is an inflection point at $x = 1$

We even have a saddle point, since $f'(1) = 0$.

(2) where are the inflection points of the function $f(x) := x^4$

$f'(x) = 4x^3$

$f''(x) = 12x^2 = 0$

candidates for inflection points: $x = 0$

$f'''(x) = 24x$

$f'''(0) = 0$

Is there a change of signs in the second derivative?

$f''(x) = 12x^2 > 0$ for all $x \neq 0$ Thus, there is no inflection point at $x = 0$