Prep Course Mathematics

Proofs

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Content

1. Methods of proof
   - Direct proof
   - Indirect proof
   - Mathematical induction

What is a proof?

Definition (Proof)

logical list of arguments, starting from a given assumption to verify (or falsify) an assertion.

⚠ As long as a statement is not proven, it may be that it is false.

Example: $F_n = 2^{2^n} + 1, \quad n \in \mathbb{N}_0$

Conjecture of Fermat (1637): all F_n are prime number.
Disproved from Euler (1732): He found 641 a real divisor of

$$F_5 = 4.294.967.297.$$

Approach:

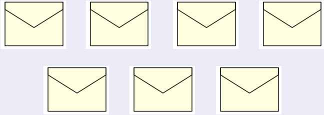
1. Understand the question: know the relevant definitions
2. Choose method of proof: similar questions known?
3. Perform the proof
4. Check: question answered, all intermediate steps correct?

Proof: When is an example enough and when not?

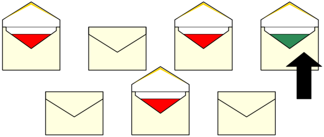
	prove	**disprove**
It-exists-statement $\exists x : A(x)$	**?**	**?**
For-all-statement $\forall x : A(x)$	**?**	**?**

Example: Prove it-exists-statement

Among the following letters
there exists one with a green card.



Proof:



We found a letter with a green card.
So the statement is proven.
It doesn't matter if there are other such letters!

Proof: When is an example enough and when not?

1. Case: Prove it-exists-statement

Suppose we have a statement of the form

„It exists an object x that fulfils $A(x)$ "

$$✎\ \exists x : A(x)$$

To prove such a statement, an **example** is enough

Reason: The statement only calls for one object, which has the desired property $A(x)$

⚠ **Attention:**

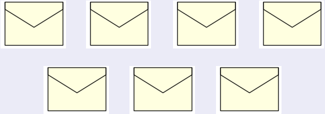
If we say „It exists ...",
then we mean: „It exists <u>at least</u> one ...".
So there could be two, three or more.

Proof: When is an example enough and when not?

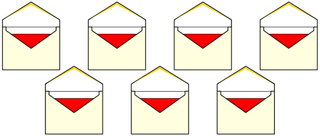
	prove	**disprove**
It-exists-statement $\exists x : A(x)$	**example**: Show that an x has the property $A(x)$.	**?**
For-all-statement $\forall x : A(x)$	**?**	**?**

Example: Prove for-all-statement

All of the following letters have a red card.



Proof:



Only when you know for each letter that there is a red card, the statement is proven! Just opening a few letters is not enough.

Proof: When is an example enough and when not?

2. Case: Prove for-all-statement

Suppose we have a statement of the form

„All objects x fulfil $A(x)$ “

$$✎\ \forall x : A(x)$$

To prove such a statement, an example is NOT enough.
A **generally valid proof** is necessary!

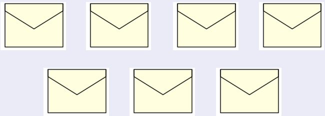
Reason: To know that an object has the property $A(x)$ does not mean that all objects have this property

Proof: When is an example enough and when not?

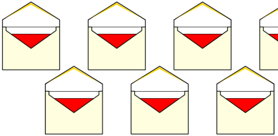
	prove	**disprove**
It-exists-statement $\exists x : A(x)$	**example**: Show that an x has the property $A(x)$.	**?**
For-all-statement $\forall x : A(x)$	**generally valid proof**: Show that all x have the property $A(x)$.	**?**

Example: Disprove it-exists-statement

Among the following letters
there exists one with a green card.



Proof:



Only when you know for each letter that there is a red card, the statement is disproved! Just opening a few letters is not enough.

Proof: When is an example enough and when not?

3. Case: Disprove it-exists-statement

Suppose we have a statement of the form

„It exists an object x that fulfils $A(x)$ “

✎ $\exists x : A(x)$

To disprove such a statement means to **prove the opposite**.
Reason: Either a statement or its opposite is true.

The opposite is a *for-all-statement*:

„All objects x do not fulfil $A(x)$ “

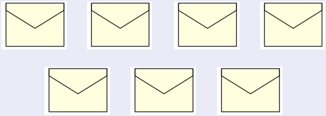
✎ $\forall x : \neg A(x)$

To prove this a **generally valid proof** is necessary!

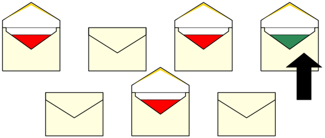
Proof: When is an example enough and when not?

	prove	**disprove**
It-exists-statement $\exists x : A(x)$	**example**: Show that an x has the property $A(x)$.	**generally valid proof**: Show that an x do $A(x)$ <u>not</u> have the property $A(x)$.
For-all-statement $\forall x : A(x)$	**generally valid proof**: Show that all x have the property $A(x)$.	**?**

Example: Disprove for-all-statement

All of the following letters have a red card. 

Proof:



We found a letter with a green card. So the statement is disproven.
It doesn't matter if there are other such letters!

Proof: When is an example enough and when not?

4. Case: Disprove for-all-statements

Suppose we have a statement of the form

„All objects x fulfil $A(x)$ "

✎ $\forall x : A(x)$

To disprove such a statement means to **prove the opposite**.
Reason: Either a statement or its opposite is true.

The opposite is an for-all-statement:

„It exists an object x that does not fulfil $A(x)$. "

✎ $\exists x : \neg A(x)$

A **(counter-)example** is enough.

Proof: When is an example enough and when not?

	prove	**disprove**
It-exists-statement $\exists x : A(x)$	**example**: Show that an x has the property $A(x)$.	**generally valid proof**: Show that an x do $A(x)$ not have the property $A(x)$.
For-all-statement $\forall x : A(x)$	**generally valid proof**: Show that all x have the property $A(x)$.	**counter example**: Show that an x does not have the property $A(x)$.

Examples

There exist natural number a, b, and c, such that $a^2 + b^2 = c^2$ holds.

Proof: (Proving It-exists-statement with an example)
For example consider $a = 3$, $b = 4$, and $c = 5$.
Then $a^2 + b^2 = 9 + 16 = 25 = c^2$.

For every real number x it holds that $x^2 - 8x + 17 \geqslant 0$.

Observation:
For $x = 1$ we have $1^2 - 8 \cdot 1 + 17 = 10 \geqslant 0$.✓
For $x = 2$ we have $2^2 - 8 \cdot 2 + 17 = 5 \geqslant 0$.✓
For $x = 3$ we have $3^2 - 8 \cdot 3 + 17 = 2 \geqslant 0$.✓
But why does the inequality hold for all $x \in \mathbb{R}$?

Proof: (Proving For-all-statement with a generally valid proof.)

Let x be a real number. Then the following holds:
$x^2 - 8x + 17 = (x - 4)^2 + 1$.
This statement is always at least 0, since both the square $(x - 4)^2$ and the summand 1 are non-negative.

Methods of proof

Direct proof

- Given: A
- Find: B
- Show that $A \implies B$, usually via $A \implies A_1 \implies A_2 \implies \ldots \implies A_n \implies B$.

Indirect proof via contraposition

- Given: A
- Find: B
- Show that $A \implies B$, by showing $\neg B \implies \neg A$.

Indirect proof via contradiction

- Show that A, by falsifying $\neg A$.

Mathematical induction

Mathematical induction

Aim: A predicate $A(n)$ should be proved for all natural numbers $n \geqslant n_0$, where $n_0 \in \mathbb{N}$.

Mathematical induction

To show that the predicate $A(n)$ is true for all $n \geqslant n_0$, can be proved as follows:

- **Base case:** Show that $A(n_0)$ is true.
- **Induction step:** Show that $A(n+1)$ is true
  under the assumption that $A(n)$ is true for some $n \geqslant n_0$.

  Short: $A(n) \Rightarrow A(n+1)$

$A(n)$ is called the induction hypothesis

Domino effect