Prep Course Mathematics

Logic

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   - Logical operations
   - Quantifiers
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Logical statement

Definition (Logical statement)

A logical statement (or proposition) is a statement, which means a meaningful declarative sentence, that is either *true or false.*

Example: Which of these are statements?

- Each natural number is also an integer.
- Please open the door.
- $3 > 7$
- How are you?
- $1 + 2$

Logical operations

Let A and B be two logical statements.

Conjunction

The statement „A and B “ ...

- is true, if both statements A and B are true.
- is false, if at least one of the statements A and B is false.

✎ $A \wedge B$

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

Example: $A :=$ The food is sweet. $B :=$ The food is sour.

$A \wedge B =$ The food is sweet **and** the food is sour.

Disjunction

The statement “A or B” ...

- is true, if at least one of the statements A and B is true.
- is false, if both statements A and B are false.

✎ $A \vee B$

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

Example: $A :=$ Today the sun is shining. $B :=$ Today it is raining.

What is the weather like, if we have $A \vee B$?

Logical operations

Implication (If ..., then ...)

The statement "A implies B" ...

- is false, if A is true and B is false.
- is true, if A is false or B is true.

✎ $A \Rightarrow B$

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

Example:

- It's raining. $\Rightarrow$ The road is wet.
- The mobile phone rings. $\Rightarrow$ The battery is not empty.

Other ways describing "$A \Rightarrow B$" in words

- From A follows B.
- A is *sufficient* for B.
- Without B, A cannot occur.
- B is *necessary* for A.
- A implies B. (implies $=$ results in)

Example: Implication

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

A grandmother makes the following promise to her grandson:

„If you pass the math test,
I'll give you a bicycle."

When is the grandma lying?
The grandma's promise is of the form $A \Rightarrow B$ with

- Statement A „The grandson passes the math test",
- Statement B: „The grandma buys a bicycle for her grandson".

The grandma is lying if A is fulfilled and B is not (line 2).
If the grandma buys a bicycle, she is not lying (line 1).
If the grandson does not pass the test, then the grandma cannot break her promise (line 3, 4).

Logical operations

Equivalence (...if and only if ...)

The statement "A is equivalent to B" ...

- is true if either both statements A and B are true or both statements A and B are false.
- is false if one of the statements A and B is true when the other is false.

✎ $A \Leftrightarrow B$

A	B	$A \Leftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

Example: The exam is passed if and only if at least 50% of the points have been achieved.

Other ways describing "$A \Leftrightarrow B$" in words

- From A follows B and from B follows A.
- A holds if and only if B holds.
- A is *sufficient* and *necessary* for B.

Exercise.

Which statements are true for all $z \in \mathbb{Z}$?

i) $z = 0 \iff z^2 = 0$

ii) $z > 0 \iff z^2 > 0$

iii) $z = \sqrt{2} \iff z = \frac{1}{2}$

Solution

i) always true since $z = 0 \Rightarrow z^2 = 0$ is true

and $z^2 = 0 \Rightarrow z = 0$ is true

ii) not always true since $z > 0 \Rightarrow z^2 > 0$ is true

but $z^2 > 0 \Rightarrow z > 0$ is not always true

iii) true, since $z = \sqrt{2}$ and $z = \frac{1}{2}$ are never true if $z \in \mathbb{Z}$

Negation

Negation

The negation of A is the statement, which ...

- is true if A is false.
- is false if A is true.

✎ $\neg A$

A	$\neg A$
T	F
F	T

Observation:
B is the negation of A if and only if both statements A and B are never true at the same time and never false at the same time.

Example:
What is the negation of the following logical statement?
$R :=$ „In this room there is a red chair."

- ✗ $A :=$ „There are no chairs in this room."
- ✗ $B :=$ „In this room there is a chair that is not red."
- ✗ $C :=$ „All chairs in this room are Green."
- ✓ $D :=$ „All chairs in this room are not red."

Negation of statement with AND/OR

Let A and B be two statements.

- The negation of the statement „A **and** B“ is „not A **or** not B“.
- The negation of the statement „A **or** B“ is „not A **and** not B“.

Reason:

A	B	$A \wedge B$ A and B	$\neg A$ not A	$\neg B$ not B	$(\neg A) \vee (\neg B)$ (not A) or (not B)
T	T	T	F	F	F
T	F	F	F	T	T
F	T	F	T	F	T
F	F	F	T	T	T

Negation

$(\neg A) \wedge (\neg B)$	$A \vee B$
F	T
F	T
F	T
T	F

Negation

Predicates and quantifiers

Definition (Predicate)

A predicate is a statement that may be true or false depending on the values of its variables.

Example: Let $A(x, y)$ be the predicate „The double of x is less than y.

- $A(2, 5)$ is true
- $A(3, 5)$ is false

Quantifiers: $\forall$ - for all $\exists$ - it exists

✎	**Meaning**
$\forall x : A(x)$	„For all x $A(x)$ is true."
$\exists x : A(x)$	„It exists at least one x for which $A(x)$ is true."
$\exists! x : A(x)$	„It exists exactly one x for which $A(x)$ is true."

⚠ If we say „It exists ...", then we mean: „It exists at least one ...".

Example:
„Every natural number is greater than 0."

✎ $\forall n \in \mathbb{N} : n > 0$

Negation of the quantifiers

A statement with quantifiers and a predicate $A(x, y, z, ...)$ is given. Then we get its **negation** by

- replacing $\forall$-quantifier by $\exists$-quantifier,
- replacing $\exists$-quantifier by $\forall$-quantifier and
- negating $A(x, y, z, ...)$.

Rule of thump

$$"\neg\forall = \exists\neg" \quad \text{and} \quad "\neg\exists = \forall\neg"$$

Example:

„It exists a natural number n so that every natural number m is at least as large as n.

✎ $\exists n \in \mathbb{N} \; \forall m \in \mathbb{N} : m \geq n$

Negation:

✎ $\forall n \in \mathbb{N} \; \exists m \in \mathbb{N} : m < n$

For every natural number n there is at least one natural number m which is smaller than n.

Example

Let A be the set of all monkeys.

$B(x)$: x likes bananas

It exists a monkey who likes bananas.

$$\exists a \in A : B(a)$$

Negation:

$$\forall a \in A : \neg B(a)$$

All monkeys do not like bananas.