# Prep Course Mathematics

## Solving equations

Dr. Simon Campese, Dr. Dennis Clemens, M. Sc. Yannick Mogge

TUHH

INSTITUTE OF MATHEMATICS

# Equations: equivalence transformations

Equivalence transformations modify equations without altering their solutions.

### Important equivalence transformations:

- swapping sides: $a = b \iff b = a$
- addition/subtraction: $a = b \iff a \pm c = b \pm c$
- multiplication with or division by *nonzero* constant $c$:

$$a = b \iff ac = bc \iff \frac{a}{c} = \frac{b}{c}$$

⚠ Taking powers or roots are **not** equivalence transformations.

# Solving equations / roots of functions

By subtracting one side, every equation in one unknown can equivalently be written in the form $f(x) = 0$.

**Example:**

$$3x + 4 = 5 \quad \iff \quad 3x - 1 = 0 \quad \iff \quad f(x) = 0 \text{ with } f(x) := 3x - 1.$$

**Important consequence:**

The solutions of an equation are precisely the roots of the associated function.

One can often infer information of the solutions from the form of the associated function.

# Linear equations

- **general form:** $ax + b = 0$ with $a \neq 0$
- **associated function:** line $\big(f(x) := ax + b\big)$
- **solutions:** exactly one: $x = -\frac{b}{a}$ where the line intersects the x-axis

# Quadratic equations

- **general form:** $ax^2 + bx + c = 0$ with $a \neq 0$
- **associated function:** parabola
- **solutions:** none, one or two
  (intersections of the parabola with the x-axis)

There are several techniques to solve quadratic equations.

# Solving quadratic equations: quadratic formula

The equation $x^2 + px + q = 0$ has roots

$$x_\pm = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}.$$

Depending on whether the radicand is negative, null or positive, there are no, one or two solutions, respectively.

**Example:** $x^2 + 2x - 8 = 0$

# Solving quadratic equations: completing the square

**Idea:** Use binomial formula to transform $ax^2 + bx + c = 0$ into $a(x - x_0)^2 + y_0 = 0$. This yields

$$x_{1,2} = x_0 \pm \sqrt{\frac{-y_0}{a}}.$$

Depending on whether the radicand is negative, null or positive, there are no, one or two solutions, respectively.

**Example:** $2x^2 - 2x - 12 = 0$

# Solving polynomial equations

How to solve equations of higher order, for example $x^5 - x - 1 = 0$?

**Two techniques** (only applicable in special cases):

- **Factoring:** Example: $x^3 + 2x^2 + x = 0$
- **Substitution:** Example: $x^4 - 10x^2 + 9 = 0$

**In general:**
There are (complicated) formulas for equations up to order four.
From order five onwards, no such formulas exist.

# Solving radical equations

In radical equations, the variable appears under one (or more) roots, and possibly outside of roots as well.

**Heuristic** to solve such equations:

1. Isolate a root under which the variable appears.
2. Take squares on both sides (this might enlarge the solution set).
3. Repeat the first two steps until all roots with variable have been eliminated.
4. Solve resulting equation.
5. Check all solution candidates to eliminate false solutions.

# Solving equations with absolute values

If absolute values appear in an equation, they can be eliminated by case-by-case analysis.

**Example:** $|x+5| = 7$