Analytic geometry and matrices

1. Determine the angle θ between the vectors $\mathbf{a} := \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{b} := \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$.

2. Calculate the area of the parallelogram whose sides are given by the following vectors $\mathbf{a}$ and $\mathbf{b}$.

   a) $\mathbf{a} := \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$, $\mathbf{b} := \begin{pmatrix} 6 \\ 11 \\ 0 \end{pmatrix}$

   b) $\mathbf{a} := \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{b} := \begin{pmatrix} 7 \\ -3 \\ 2 \end{pmatrix}$

3. Let the following points be given:

$$\mathbf{a} := \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}, \ \mathbf{b} := \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \ \mathbf{c} := \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix}$$

   a) Determine a parameter representation for the plane E containing the 3 points $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

   b) Determine a parameter representation for the straight line g that passes through $\mathbf{a}$ and is perpendicular to E.

4. a) Let E be the plane in $\mathbb{R}^3$ passing through the point P with location vector $\mathbf{p} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and which is perpendicular to the vector $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$. Determine a normal form and coordinate form of E.

   b) Let $\mathbf{p} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and E be the plane in $\mathbb{R}^3$ given by

$$E := \left\{ \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} : \lambda, \mu \in \mathbb{R} \right\}.$$

   Determine a normal form of E and check whether the point $\mathbf{p}$ lies in E. Determine the distance from $\mathbf{p}$ to E.

5. Find α and β such that $\mathbf{v} = \alpha\mathbf{u} + \beta\mathbf{w}$ where $\mathbf{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, and

   a) $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

   b) $\mathbf{v} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

   c) $\mathbf{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

   d) $\mathbf{v} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

6. Consider $\mathbf{A} = \begin{pmatrix} 1 & 4 & 3 \\ 3 & 2 & 1 \\ 5 & 5 & a \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 58 \\ 40 \\ 81 \end{pmatrix}$. Determine for which values of a the linear system of equations $\mathbf{Ax} = \mathbf{b}$ has:

   - exactly one solution.
   - none or infinitely many solutions.

7. Consider the map $f_{\mathbf{A}} : \mathbb{R}^2 \to \mathbb{R}^2$, $f_{\mathbf{A}}(\mathbf{x}) = \mathbf{Ax}$, where

   a) $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

   b) $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

   c) $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$

   Sketch the pointwise application of the linear mapping $f_{\mathbf{A}}$ to the house: