Methods of proof

1. a) Let $n \in \mathbb{Z}$ be an even number. That means there is a number $p \in \mathbb{Z}$ such that $n = 2p$. So

$$n \cdot m = 2p \cdot m = 2r,$$

where $r := p \cdot m$ is again an integer. Thus $n \cdot m$ is an even number.

b) Proof direction "'$\Leftarrow$"': Let $n \in \mathbb{Z}$ be an odd number. That means there exists a number $p \in \mathbb{Z}$ such that $n = 2p + 1$. Thus

$$n + 1 = 2p + 1 + 1 = 2p + 2 = 2(p + 1).$$

There is a number $q \in \mathbb{Z}$, namely $q := p + 1$ such that $n + 1 = 2q$. It follows that $n + 1$ is even.
Direction of proof "'$\rightarrow$"': Let $n \in \mathbb{Z}$ and $n + 1$ be an even number. That means there exists a number $p \in \mathbb{Z}$ such that $n + 1 = 2p$. So

$$n = n + 1 - 1 = 2p - 1 = 2p - 2 + 1 = 2(p - 1) + 1 = 2q + 1,$$

where $q := p - 1$ with $q \in \mathbb{Z}$. Thus $n = 2q + 1$ is an odd number.

c) Let $n \in \mathbb{Z}$ be an even number. That means there is a number $p \in \mathbb{Z}$ such that $n = 2p$. It follows that:

$$n^2 = (2p)^2 = 2p \cdot 2p = 2\left(2p^2\right).$$

Thus $n^2 = 2m$ holds with $m := 2p^2 \in \mathbb{Z}$, since $p \in \mathbb{Z}$. Consequently, n^2 is even.
Alternatively, one can use statement (a) for $n^2 = n \cdot n$.

2. a) **Induction start ($n = 1$):** For $n = 1$ the relevant term is

$$7^n - 4^n = 7 - 4 = 3\ ,$$

which is divisible by 3.

Induction assumption: We assume that we have already shown the statement for n, that is, we assume that $7^n - 4^n$ is divisible by 3.

Induction step ($n \to n + 1$): Our goal now is to prove the statement for $n + 1$ under the induction assumption. That is, we show that $7^{n+1} - 4^{n+1}$ is divisible by 3. Clever transformation yields

$$\begin{aligned} 7^{n+1} - 4^{n+1} &= 7 \cdot 7^n - 4 \cdot 4^n \\ &= 6 \cdot 7^n + 7^n - 3 \cdot 4^n - 4^n \\ &= 3 \cdot (2 \cdot 7^n - 4^n) + (7^n - 4^n)\ . \end{aligned}$$

Since $3 \cdot (2 \cdot 7^n - 4^n)$ is divisible by 3 and, by induction assumption, $7^n - 4^n$ is also divisible by 3, it follows that their sum, i.e. $7^{n+1} - 4^{n+1}$, is also divisible by 3.

b) **Induction start ($n = 5$):** For $n = 5$ the inequality is

$$32 > 25 \ ,$$

which is correct.

induction assumption: We assume that we have already shown the inequality for n, that is, $2^n > n^2$ holds.

induction step ($n \to n+1$): Our goal now is to prove the statement for $n+1$ under the induction assumption. That is, we show $2^{n+1} > (n+1)^2$.

First of all, according to induction assumption (IV), it holds that

$$2^{n+1} = 2 \cdot 2^n \overset{\text{IV}}{>} 2 \cdot n^2 \ .$$

Thus, if we manage to show that $2 \cdot n^2 \geq (n+1)^2$ holds, then it follows that $2^{n+1} > (n+1)^2$ (the statement for $n+1$). To show the inequality $2 \cdot n^2 \geq (n+1)^2$, we apply äquivalence transformations:

$$\begin{aligned} 2 \cdot n^2 \geq (n+1)^2 &\Leftrightarrow\ 2n^2 \geq n^2 + 2n + 1 \\ &\Leftrightarrow\ n^2 - 2n - 1 \geq 0 \\ &\Leftrightarrow\ (n-1)^2 - 2 \geq 0 \ . \end{aligned}$$

The last inequality is true, since $n \geq 5$ is assumed. and thus $(n-1)^2 - 2 \geq 14$ holds.