Prep Course Mathematics

Vectors and systems of linear equations

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Content

What is a vector?

In physics we distinguish scalar and vector quantities.

Scalar quantities are characterized by a **numerical value** (with unit).

Vector quantities are given by a **numerical value** (with unit) and a **direction**.

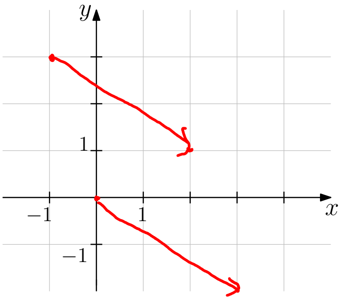
Vector quantities are often represented by an arrow (vector) the length of which represents the numerical value.

The **Analytic Geometry** uses vectors for the description of e.g.

- geometric objects (lines, planes, triangles etc.)
- geometric operations (rotations, reflections etc.)

Vectors in the plane

Plane: two coordinate axes (e.g. x- and y-axis) intersecting in the zero-point (origin)

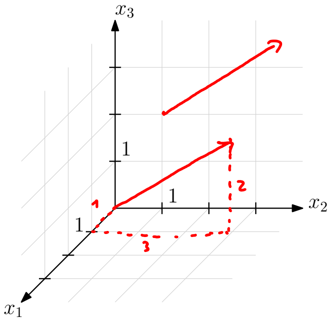


A vector in this plane is an arrow of which only the elongation in the direction of the x- and y-axes is known.

Example: $\mathbf{v} := \binom{3}{-2}$ is a vector which starts at an arbitrary point and goes 3 steps in the direction of the x-axis and -2 steps in the direction of the y-axis.

Vectors in 3-dimensionsal space

3-dim space: three coordinate axes (e.g. x_1-, x_2- and x_3-axis) intersecting in the zero-point (origin)



A vector in this space is an arrow of which only the elongation in the direction of the x_1-, x_2- and x_3-axes is known.

Example: $\mathbf{v} := \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ goes 1 step in the direction of the x_1-axis, 3 steps in the direction of the x_2-axis and 2 step in the direction of the x_3-axis.

Vectors in $\mathbb{R}^n$

Vectors in $\mathbb{R}^n$

Let $n \in \mathbb{N}$. Every object $\mathbf{v}$ of the form

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \quad \text{mit } v_1, v_2, \ldots, v_n \in \mathbb{R}$$

is called a real vector.

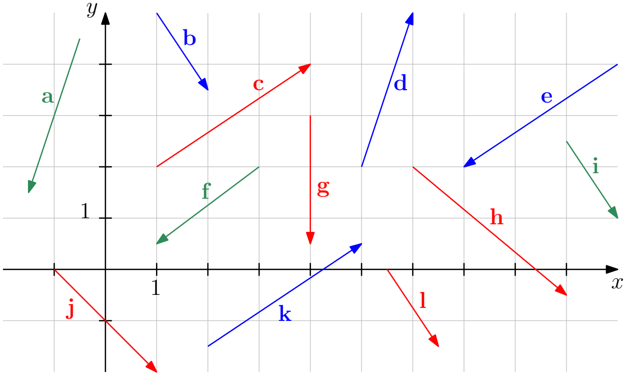
The set of all such vectors is denoted by $\mathbb{R}^n$.

The entries $v_1, v_2, \ldots, v_n$ are called components.

Note:

- *Today we mainly consider the case $n \in \{2, 3\}$. However, in Linear Algebra we will often work with larger $n \in \mathbb{N}$.*
- *In different lectures you may see different notation for vectors: $\mathbf{v}$, $\vec{v}$, $\underline{v}$ etc.*

Exercise

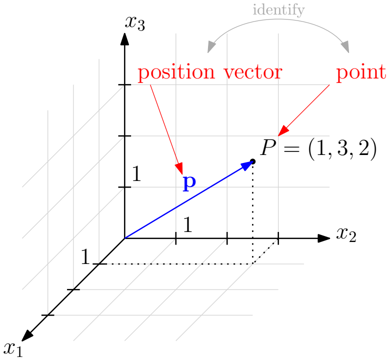


Exercise: Which of the given vectors are equal? Give their representations with coordinates.

$k = c = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ $\qquad$ $k = -e$

$b = i = l = \begin{pmatrix} 1 \\ -1.5 \end{pmatrix}$ $\qquad$ $a = -d$

Points and position vectors



Definition (position vector)

Let O be the origin (of the coordinate system). If P is a point, then the vector going from O to P is called the position vector of P. We often identify points with their position vectors.

Vectors between two points

Vectors between two points

Let A and B be two points. The vector $\mathbf{v}_{AB}$ going from A to B can be obtained by subtracting the position vector of A from the position vector of B.

$$\mathbf{v}_{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ \vdots \end{pmatrix}$$

$$B = (b_1, b_2, \dots)$$

$$A = (a_1, a_2, \dots)$$

Example: The vector going from $A := (2, 4, -6)$ to $B := (3, -1, 9)$ is

$$\mathbf{v}_{AB} = \begin{pmatrix} 3 - 2 \\ -1 - 4 \\ 9 - (-6) \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 15 \end{pmatrix}$$

Length of a vector

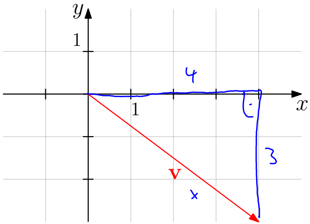
Length of a vector

Consider any vector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$. Then the length (norm) of this vector is

$$\|\mathbf{v}\| := \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}\,.$$

Example: The length of the vector $\mathbf{v} := \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ is ... $\|v\| = \sqrt{4^2 + (-3)^2}$

$= \sqrt{25} = 5$



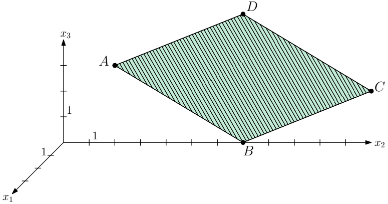
$x^2 = 4^2 + 3^2 = 25$

$x = \sqrt{25} = 5$

Exercise

Consider a parallelogramm in the 3-dimensional space which has the corner points A, B, C, D of which only three are given:

$A := (2,3,4)$, $B := (0,7,0)$ and $D := (-2,6,4)$.



(i) Determine the vectors $\mathbf{v}_{AB}$ and $\mathbf{v}_{AD}$ which lead from A to B respectively D.

(ii) Which coordinates does the point C have?

(iii) Determine the circumference of the parallelogramm.

(iv) Determine the length of the diagonal AC.

(i) $v_{AB} = \begin{pmatrix} 0-2 \\ 7-3 \\ 0-4 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ -4 \end{pmatrix}$ $\qquad v_{AD} = \begin{pmatrix} -2-2 \\ 6-3 \\ 4-4 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix}$

(ii) Since we have a parallelogramm, $v_{AD} = v_{BC}$

with $v_{BC} = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix}$ we go from $B = \begin{pmatrix} 0 \\ 7 \\ 0 \end{pmatrix}$ to C

Hence, $C = (0-4,\ 7+3,\ 0+0) = (-4, 10, 0)$

(iii) $\|v_{AB}\| = \sqrt{(-2)^2 + 4^2 + (-4)^2} = \sqrt{4 + 16 + 16} = 6$

$\|v_{AD}\| = \sqrt{(-4)^2 + 3^2 + 0^2} = \sqrt{16 + 9} = 5$

Circumference is $6 + 5 + 6 + 5 = 22$

(iv) Since $A = (2, 3, 4)$ and $C = (-4, 10, 0)$ we have $v_{AC} = \begin{pmatrix} -4 - 2 \\ 10 - 3 \\ 0 - 4 \end{pmatrix} = \begin{pmatrix} -6 \\ 7 \\ -4 \end{pmatrix}$

Hence, $\|v_{AC}\| = \sqrt{(-6)^2 + 7^2 + (-4)^2} = \sqrt{101}$

Addition/Subtraction of vectors

Addition/Subtraction of vectors in $\mathbb{R}^n$

The addition/subtraction of two vectors in $\mathbb{R}^n$ is done componentwisely.
Formally: Consider two vectors

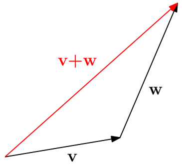
$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}, \ \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} \in \mathbb{R}^n .$$

Then we define

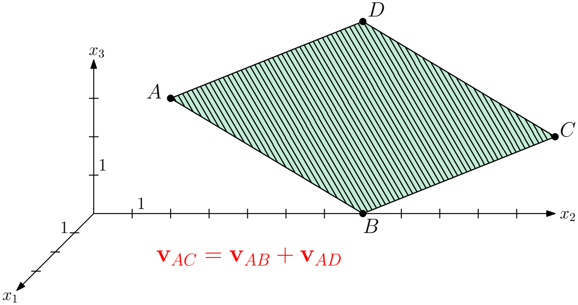
$$\mathbf{v} + \mathbf{w} := \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{pmatrix} \quad \text{and} \quad \mathbf{v} - \mathbf{w} := \begin{pmatrix} v_1 - w_1 \\ v_2 - w_2 \\ \vdots \\ v_n - w_n \end{pmatrix} .$$

Addition/Subtraction of vectors

Addition of vectors: The addition of vectors corresponds to a composition of movements that are described by the vectors.



Example:



$$\mathbf{v}_{AC} = \mathbf{v}_{AB} + \mathbf{v}_{AD}$$

Scalar multiplication of vectors

Scalar multiplication of vectors in $\mathbb{R}^n$

The scalar multiplication of vectors in $\mathbb{R}^n$ is done componentwisely.
Formally: Consider any real number $\lambda \in \mathbb{R}$ and any vector

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n \,.$$

Then we define

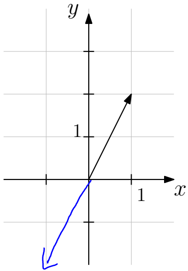
$$\lambda \cdot \mathbf{v} := \begin{pmatrix} \lambda \cdot v_1 \\ \lambda \cdot v_2 \\ \vdots \\ \lambda \cdot v_n \end{pmatrix} \,.$$

$$5 \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \cdot 1 \\ 5 \cdot 2 \\ 5 \cdot 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ 15 \end{pmatrix}$$

Scalar multiplication of vectors

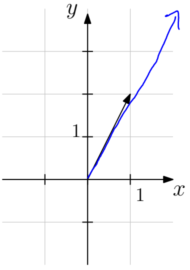
Examples: Consider the vector $\mathbf{v} = \binom{1}{2}$. We determine and draw the following three vectors:

(i) $-1 \cdot \mathbf{v} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$



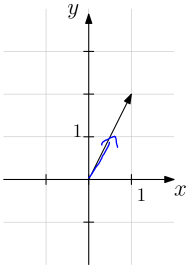
direction is switched

(ii) $2 \cdot \mathbf{v} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$



stretching

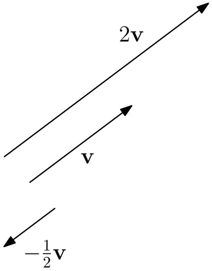
(iii) $\frac{1}{2} \cdot \mathbf{v} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$



compression

Scalar multiplication of vectors

scalar multiplication for vectors: The scalar multiplication of a vector with a constant α corresponds to a stretching by the factor $|\alpha|$, while the direction is switched if α is negative.



Examples

$$3 \cdot \begin{pmatrix} 13 \\ -12 \end{pmatrix} - 2 \cdot \begin{pmatrix} 5 \\ -4 \end{pmatrix} = \begin{pmatrix} 39 \\ -36 \end{pmatrix} - \begin{pmatrix} 10 \\ -8 \end{pmatrix} = \begin{pmatrix} 29 \\ -28 \end{pmatrix}$$

$$2 \cdot \left(\underbrace{\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 5 \\ -5 \end{pmatrix}}_{\begin{pmatrix} 3 \\ 7 \\ -6 \end{pmatrix}} \right) - \begin{pmatrix} 6 \\ -2 \\ 2 \end{pmatrix} = 2 \cdot \begin{pmatrix} 3 \\ 7 \\ -6 \end{pmatrix} - \begin{pmatrix} 6 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 14 \\ -12 \end{pmatrix} - \begin{pmatrix} 6 \\ -2 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 16 \\ -14 \end{pmatrix}$$

Rules of calculation

Rules of calculation in $\mathbb{R}^n$

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ be vectors and $\alpha, \beta \in \mathbb{R}$ be numbers. Then the following hold:

(i) Commutativity: $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$

(ii) Associativity: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(iii) Distributivity I: $\alpha \cdot (\mathbf{u} + \mathbf{v}) = \alpha \cdot \mathbf{u} + \alpha \cdot \mathbf{v}$

(iv) Distributivity II: $(\alpha + \beta) \cdot \mathbf{v} = \alpha \cdot \mathbf{v} + \beta \cdot \mathbf{v}$

Exercise

(a) Calculate!

(i) $4 \cdot \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} - 5 \cdot \begin{pmatrix} -3 \\ 8 \\ -2 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 \\ 3 \\ -5 \end{pmatrix} - \left(\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \right)$

Solution:

(i) $4 \cdot \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} - 5 \begin{pmatrix} -3 \\ 8 \\ -2 \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ -8 \end{pmatrix} - \begin{pmatrix} -15 \\ 40 \\ -10 \end{pmatrix} = \begin{pmatrix} 27 \\ -40 \\ 2 \end{pmatrix}$

(ii) $\begin{pmatrix} 0 \\ 3 \\ -5 \end{pmatrix} - \left(\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 3 \\ -5 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ -4 \end{pmatrix}$

Exercise

(b) Find a vector $\mathbf{x}$ which satisfies the given equation!

(i) $2\mathbf{x} - 3 \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 4\mathbf{x} + \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix}$ (ii) $2 \cdot \left(\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} - \mathbf{x} \right) = \mathbf{x} + \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$

Solution:

(i)

$$2x - 3\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 4x + \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix}$$

$$\Leftrightarrow 2x - \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} = 4x + \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix} \qquad \Big| -4x$$

$$\Leftrightarrow -2x - \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix} \qquad \Big| +\begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow -2x = \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix} \qquad \Big| \cdot \left(-\tfrac{1}{2}\right)$$

$$\Leftrightarrow x = \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix}$$

(ii)

$$2 \cdot \left(\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} - x \right) = x + \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 6 \\ 2 \\ 2 \end{pmatrix} - 2x = x + \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \qquad \Big| -x$$

$$\Leftrightarrow \begin{pmatrix} 6 \\ 2 \\ 2 \end{pmatrix} - 3x = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \qquad \Big| -\begin{pmatrix} 6 \\ 2 \\ 2 \end{pmatrix}$$

$$\Leftrightarrow -3x = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 6 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -6 \\ 1 \\ -1 \end{pmatrix} \qquad \Big| \cdot \left(-\tfrac{1}{3}\right)$$

$$\Leftrightarrow x = \begin{pmatrix} 2 \\ -\frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

Exercise

(c) Find numbers $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that the following equation is true!

$$\alpha_1 \cdot \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \alpha_3 \cdot \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -7 \\ -1 \end{pmatrix}$$

Solution:

$$\begin{array}{l} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \end{array} \begin{pmatrix} 2\alpha_1 & + \alpha_2 & + 4\alpha_3 \\ & 3\alpha_2 & - 2\alpha_3 \\ & & - 2\alpha_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -7 \\ -1 \end{pmatrix}$$

III $\quad -2\alpha_3 = -1 \Rightarrow \alpha_3 = \frac{1}{2}$

II $\quad 3\alpha_2 - 2\alpha_3 = -7 \Rightarrow 3\alpha_2 - 1 = -7 \Rightarrow 3\alpha_2 = -6 \Rightarrow \alpha_2 = -2$

I $\quad 2\alpha_1 + \alpha_2 + 4\alpha_3 = 3 \Rightarrow 2\alpha_1 - 2 + 2 = 3 \Rightarrow \alpha_1 = \frac{3}{2}$

$\alpha_1 = \frac{3}{2}, \quad \alpha_2 = -2, \quad \alpha_3 = \frac{1}{2}$

Linear combination

Linear combination

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ and $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}$ be given. Then we call the vector

$$\mathbf{v} = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \ldots + \lambda_k \mathbf{v}_k$$

a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.

We also say that $\mathbf{v}$ can be generated by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.

Example:

$$\underbrace{8 \cdot \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + 7 \cdot \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}}$$

linear combination of $\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

$$\frac{3}{2} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} - 2 \cdot \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -7 \\ -1 \end{pmatrix}$$

$\Rightarrow$ $\begin{pmatrix} 3 \\ -7 \\ -1 \end{pmatrix}$ can be written as a linear combination of $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix}$

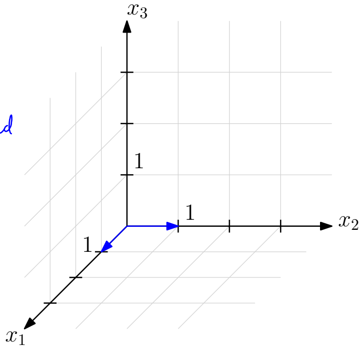
Linear combination

Example: Every point (position vector) $\mathbf{x} \in \mathbb{R}^3$ lying in the x_1x_2-plane can be generated by the vectors

$$\mathbf{e}_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{e}_2 := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

$\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = 2 \cdot e_1 + 3 e_2$

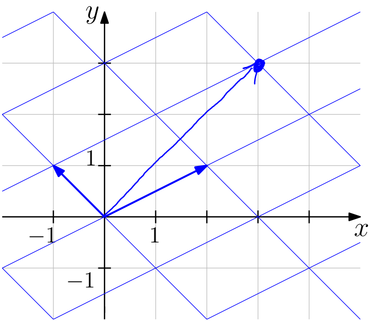
$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ cannot be generated by e_1 and e_2.



Example

Every point (position vector) $\mathbf{x} \in \mathbb{R}^2$ can be generated by the vectors

$$\mathbf{v}_1 := \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \ \mathbf{v}_2 := \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$



$\begin{pmatrix} 3 \\ 3 \end{pmatrix} = 2 \cdot v_1 + v_2$

$\Rightarrow \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ is a linear combination of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Span

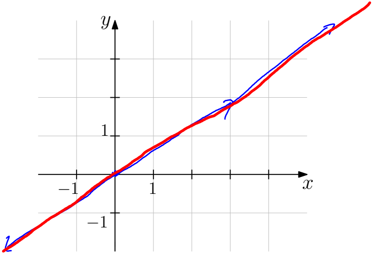
Span

Consider vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$.

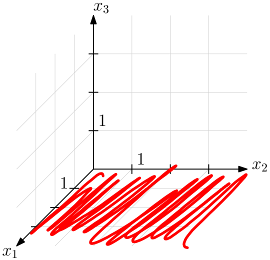
The set of all linear combinations of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ is called span (or linear hull) of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. Shortly:

$$\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) := \{\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \ldots + \lambda_k \mathbf{v}_k : \ \lambda_1, \ldots, \lambda_k \in \mathbb{R}\} \ .$$

$$\mathrm{Span}\left(\begin{pmatrix} 3 \\ 2 \end{pmatrix} \right)$$



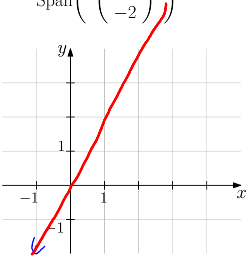
$$\mathrm{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right)$$



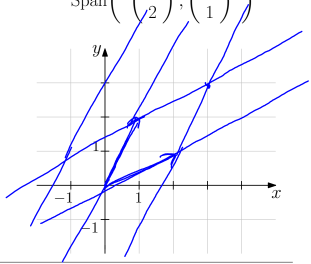
Exercise

Sketch the following spans!

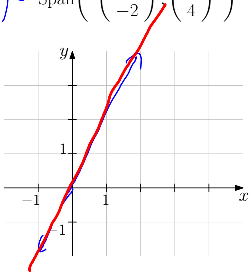
$\mathrm{Span}\left(\begin{pmatrix} -1 \\ -2 \end{pmatrix} \right)$



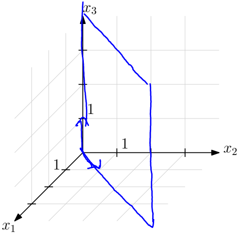
$\mathrm{Span}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right)$



$\mathrm{Span}\left(\begin{pmatrix} -1 \\ -2 \end{pmatrix} \right) = \mathrm{Span}\left(\begin{pmatrix} -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right)$



$\mathrm{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$



Linear combination $\to$ system of linear equations

The following question appears frequently in Linear Algebra:

"Given a fixed vector $\mathbf{v}$, can it be written as linear combination of some other given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots$?"

$$\begin{pmatrix} 8 \\ 17 \\ 0 \end{pmatrix} = \boxed{x_1} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \boxed{x_2} \begin{pmatrix} -3 \\ -5 \\ 5 \end{pmatrix} + \boxed{x_3} \begin{pmatrix} 5 \\ 9 \\ -4 \end{pmatrix}$$

This corresponds to asking whether a certain *system of linear equations* has a solution:

$$\begin{array}{rcrcrcr} 8 & = & 1x_1 & - & 3x_2 & + & 5x_3 \\ 17 & = & 2x_1 & - & 5x_2 & + & 9x_3 \\ 0 & = & -1x_1 & + & 5x_2 & - & 4x_3 \end{array}$$

System of linear equations

System of linear equations

Let $m, n \in \mathbb{N}$. A system of linear equations (LES) in the variables $x_1, x_2, \ldots, x_n$ is of the form

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}$$

with a_{ij} and b_i being (usually real) numbers. An assignment of values for $x_1, \ldots, x_n$ such that all equations are satisfied is called a solution of this system of equations. Such a solution is written as a vector.

System of linear equations

Example: The system

$$\begin{array}{rcrcrcr} 1x_1 & - & 3x_2 & + & 5x_3 & = & 8 \\ 2x_1 & - & 5x_2 & + & 9x_3 & = & 17 \\ -1x_1 & + & 5x_2 & - & 4x_3 & = & 0 \end{array}$$

is a system of linear equations with 3 equations and 3 variables.

A solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 2 \end{pmatrix} .$$

Method of equalization

Method of equalization

In the method of equalization, we isolate the same variable in 2 equations and equal the obtained expressions. By this, we obtain an equation which does not use one of the given variables.

Example: We solve the following system with the method of equalization:

$$\begin{array}{rcrcr} 2x_1 & - & 4x_2 & = & 15 \\ -4x_1 & + & 2x_2 & = & -9 \end{array}$$

Both equations are solved for the same variable: $x_1 = \frac{15}{2} + 2x_2$

$x_1 = \frac{9}{4} + \frac{1}{2}x_2$

Both solved terms are set equal to each other, and the resulting equation is then solved: $\frac{15}{2} + 2x_2 = \frac{9}{4} + \frac{1}{2}x_2 \Rightarrow x_2 = -3{,}5$

This result can be put into either of the two original equations, in order to obtain the value for the second variable: $x_1 = \frac{15}{2} + 2\cdot(-3.5) = 0{,}5$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.5 \\ -3.5 \end{pmatrix}$

Method of substitution

Method of substitution

In the method of substitution, we isolate some variable in an equation and replace it in the other equations. By this, we obtain equations which do not use one of the given variables.

Example: We solve the following system with the method of substitution:

$$\begin{array}{rcrcr} 2x_1 & - & 4x_2 & = & 15 \\ -4x_1 & + & 2x_2 & = & -9 \end{array}$$

One of the two equations will be solved for one of the two variables:

$$x_1 = \frac{15}{2} + 2x_2$$

We put the solved expression into the second equation and solve:

$$-4\left(\frac{15}{2} + 2x_2\right) + 2x_2 = -9$$

$$\Leftrightarrow -30 - 8x_2 + 2x_2 = -9$$

$$\Leftrightarrow -6x_2 = 21$$

$$\Leftrightarrow x_2 = -3{,}5$$

Take this result and put it back into step 1: $x_1 = \frac{15}{2} + 2(-3.5) = 0.5$ $\quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.5 \\ -3.5 \end{pmatrix}$

Exercise

From II it follows: $x_1 = -2x_2 + 2$

Then we have $3(-2x_2 + 2) + 5x_2 = 7 \iff -6x_2 + 6 + 5x_2 = 7$

$\iff x_2 = -1$

Hence, $x_1 = -2 \cdot (-1) + 2 = 4$ $\quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

(a) Solve the following system with the method of substitution:

$$\begin{array}{lrcrcr} \text{I} & 3x_1 & + & 5x_2 & = & 7 \\ \text{II} & 2x_1 & + & 4x_2 & = & 4\,. \end{array}$$

(b) Solve the following system with the method of equalization:

$$\begin{array}{lrcrcrl} \text{I}^{*} & 4x_1 & + & 8x_2 & = & 14 & \iff x_1 = -2x_2 + 3.5 \\ \text{II}^{*} & 2x_1 & - & x_2 & = & -3\,. & \iff x_1 = 0.5x_2 - 1.5 \end{array}$$

Hence, we have $-2x_2 + 3.5 = 0.5x_2 - 1.5$

$\iff -2.5x_2 = -5$

$\iff x_2 = 2$

Thus, $x_1 = 0.5 \cdot 2 - 1.5 = -0.5$ $\quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -0.5 \\ 2 \end{pmatrix}$

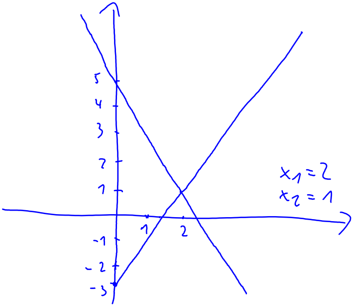
Solving by graphing

Solving by graphing (2 variables)

Consider a system of linear equations with 2 variables. Then each equation can be represented by a line in a 2-dimensional coordinate system. The set of solutions then is represented by the intersection of all lines.

Example: We solve the following system by graphing:

$$\begin{array}{rcrcl} 4x_1 & - & 2x_2 & = & 6 \\ 2x_1 & + & x_2 & = & 5 \end{array} \quad \begin{array}{l} \Leftrightarrow x_2 = 2x_1 - 3 \\ \Leftrightarrow x_2 = -2x_1 + 5 \end{array}$$

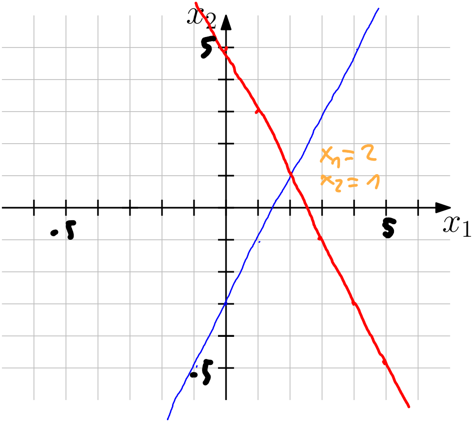


$x_1 = 2$
$x_2 = 1$

Solving by graphing

Example: We solve the following system by graphing:

$$\begin{array}{rcrcl} 4x_1 & - & 2x_2 & = & 6 \\ 2x_1 & + & x_2 & = & 5 \end{array}$$



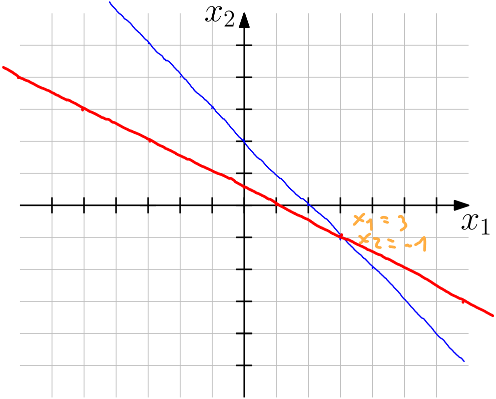
Notes:

- $x_2 = 2x - 3$
- $x_2 = -2x + 5$

Exercise

Solve the following system by graphing:

$$\begin{array}{lrcrcr} \bullet & 3x_1 & + & 3x_2 & = & 6 \\ \bullet & 2x_1 & + & 4x_2 & = & 2 \end{array}$$

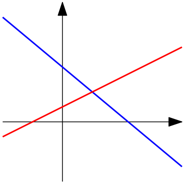


Notes:

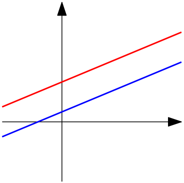
- $x_2 = -x_1 + 2$
- $x_2 = -\frac{1}{2}x_1 + \frac{1}{2}$

Solvability: number of solutions

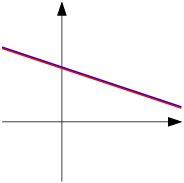
For any lines there exist three kinds of intersections:



intersection point
(one solution)



parallel
(no solution)



identical
(infinitely many solutions)

In general the following is true:

Solvability of systems of linear equations

Every system of linear equations has either (i) exactly one solution or (ii) no solution or (iii) infinitely many solutions.

Method of elimination

Method of elimination

In the method of elimination two equations (or multiples of them) are added/subtracted such that at least one variable is eliminated.

Example 1: We solve the following system with the method of elimination:

$$\begin{array}{lrcrcr} \mathrm{I} & 2x_1 & - & 4x_2 & = & 15 \\ \mathrm{II} & -4x_1 & + & 2x_2 & = & -9 \end{array} \quad \Big| + 2\cdot\mathrm{I}$$

$$\begin{array}{rcrcr} 2x_1 & - & 4x_2 & = & 15 \\ 0x_1 & - & 6x_2 & = & 21 \end{array}$$

From the second row it follows, that $x_2 = -3.5$

Consequently, from the first row it follows: $2x_1 - 4\cdot(-3.5) = 15$

$$\Rightarrow x_1 = 0.5$$

Method of elimination

Method of elimination

In the method of elimination two equations (or multiples of them) are added/subtracted such that at least one variable is eliminated.

Example 2: We solve the following system with the method of elimination:

$$\begin{array}{lrcrcr} \mathrm{I} & 4x_1 & - & 5x_2 & = & 2 \\ \mathrm{II} & -8x_1 & + & 10x_2 & = & -4 \quad \mid +2\cdot \mathrm{I} \end{array}$$

$$\begin{array}{rcrcr} 4x_1 & - & 5x_2 & = & 2 \\ 0x_1 & + & 0x_2 & = & 0 \end{array}$$

Hence, all points on the line $4x_1 - 5x_2 = 2$ are solutions.

$$x_2 = \frac{4x_1}{5} - \frac{2}{5}$$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ with $x_2 = \frac{4x_1}{5} - \frac{2}{5}$ fulfil the LES.

Exercise

Solve the following systems with the method of elimination:

(a) $$\begin{array}{rcrcr} 1x_1 & - & 4x_2 & = & 1 \\ 2x_1 & - & 2x_2 & = & 3 \end{array}$$

(b) $$\begin{array}{rcrcr} -3x_1 & + & 2x_2 & = & 5 \\ 9x_1 & - & 6x_2 & = & 13 \end{array}$$ $| + 3 \cdot \mathrm{I}$

Solution:

a) $1x_1 - 4x_2 = 1$
$2x_1 - 2x_2 = 3 \quad | - 2 \cdot \mathrm{I}$

$1x_1 - 4x_2 = 1$
$0x_1 + 6x_2 = 1$

$\Rightarrow x_2 = \frac{1}{6}$

and $x_1 - 4 \cdot \frac{1}{6} = 1$

$\Leftrightarrow x_1 = \frac{10}{6} = \frac{5}{3}$

b) $-3x_1 + 2x_2 = 5$
$0x_1 + 0x_2 = 28$ $\lightning$ no solution

This LES has no solution.

Solving bigger systems of linear equations

Observations

- The set of solutions does not change if the ordering of the equations is changed.
- The set of solutions does not change if one equation is subtracted (multiple times) from the other equations.

Recipe for calculations:

- Bring the system to a "triangle form" or "row echelon form".
- Afterwards, determine the set of solutions.

Example

We determine the unique solution of the following system of linear equations:

$$
\begin{array}{lrcrcrcrl}
\mathrm{I} & 1x_1 & + & 3x_2 & - & 3x_3 & = & 3 & \\
\mathrm{II} & 2x_1 & + & 7x_2 & - & 5x_3 & = & 4 & \mid -2\cdot\mathrm{I} \\
\mathrm{III} & -1x_1 & + & 1x_2 & + & 9x_3 & = & -13 & \mid +1\cdot\mathrm{I}
\end{array}
$$

$$
\begin{array}{rcrcrcrl}
1\cdot x_1 & + & 3x_2 & - & 3x_3 & = & 3 & \\
0\cdot x_1 & + & 1x_2 & + & 1x_3 & = & -2 & \\
0\cdot x_1 & + & 4x_2 & + & 6x_3 & = & -10 & \mid -4\cdot\mathrm{II}
\end{array}
$$

$$
\begin{array}{rcrcrcr}
1x_1 & + & 3x_2 & - & 3x_3 & = & 3 \\
0\cdot x_1 & + & 1x_2 & + & 1x_3 & = & -2 \\
0\cdot x_1 & + & 0x_2 & + & 2x_3 & = & -2
\end{array}
$$

$\mathrm{III}: \; 2x_3 = -2 \Rightarrow x_3 = -1$

$\mathrm{II}: \; 1\cdot x_2 + 1\cdot(-1) = -2 \Rightarrow x_2 = -1$

$\mathrm{I}: \; 1\cdot x_1 + 3\cdot(-1) - 3\cdot(-1) = 3 \Rightarrow x_1 = 3$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$$

Exercise

Determine the unique solution of the following system of linear equations:

$$
\begin{array}{lrcrcrcrl}
\mathrm{I} & 1x_1 & + & 3x_2 & - & 2x_3 & = & 5 & \\
\mathrm{II} & 3x_1 & + & 11x_2 & - & 5x_3 & = & 11 & \mid -3\cdot \mathrm{I} \\
\mathrm{III} & 2x_1 & + & 2x_2 & - & 4x_3 & = & 14 & \mid -2\cdot \mathrm{I} \\
\hline
 & 1x_1 & + & 3x_2 & - & 2x_3 & = & 5 & \\
 & 0x_1 & + & 2x_2 & + & 1x_3 & = & -4 & \\
 & 0x_1 & - & 4x_2 & + & 0x_3 & = & 4 &
\end{array}
$$

$\mathrm{III}: \quad x_2 = -1$

$\mathrm{II}: \quad 2\cdot(-1) + x_3 = -4 \quad \Rightarrow \quad x_3 = -2$

$\mathrm{I}: \quad x_1 + 3\cdot(-1) - 2\cdot(-2) = 5 \quad \Rightarrow \quad x_1 = 4$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ -2 \end{pmatrix}$$

An example with infinitely many solutions

We determine the solutions of the following system of linear equations:

$$\begin{array}{lrcrcrcr} \mathrm{I} & 1x_1 & - & 5x_2 & + & 4x_3 & = & -2 \\ \mathrm{II} & 1x_1 & - & 4x_2 & + & 2x_3 & = & 1 \\ \mathrm{III} & -2x_1 & + & 7x_2 & - & 2x_3 & = & -5 \end{array} \quad \begin{array}{l} \\ \mid -1\cdot \mathrm{I} \\ \mid +2\cdot \mathrm{I} \end{array}$$

$$\begin{array}{rcrcrcr} 1x_1 & - & 5x_2 & + & 4x_3 & = & -2 \\ 0x_1 & + & 1x_2 & - & 2x_3 & = & 3 \\ 0x_1 & - & 3x_2 & + & 6x_3 & = & -9 \end{array} \quad \begin{array}{l} \\ \\ \mid +3\cdot \mathrm{II} \end{array}$$

$$\begin{array}{rcrcrcr} 1x_1 & - & 5x_2 & + & 4x_3 & = & -2 \\ 0x_1 & + & 1x_2 & - & 2x_3 & = & 3 \\ (0x_1 & + & 0x_2 & + & 0x_3 & = & 0) \end{array}$$

$\mathrm{II}: \quad x_2 = 3 + 2x_3$

$\mathrm{I}: \quad x_1 - 5\cdot(3+2x_3) + 4x_3 = -2 \implies x_1 - 15 - 10x_3 + 4x_3 = -2$

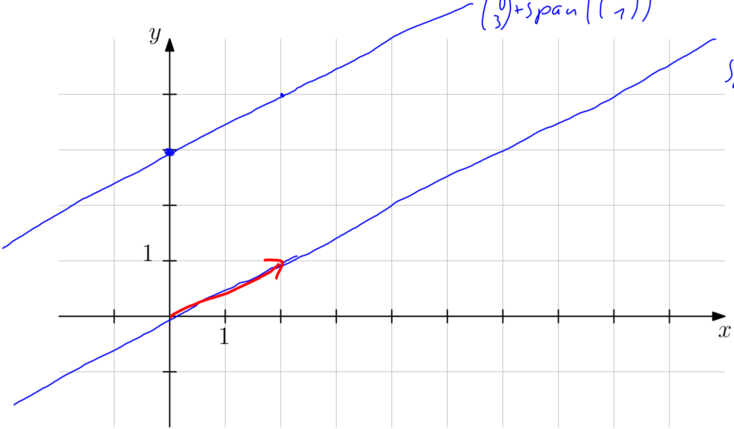
$\implies x_1 = 13 + 6x_3$

x_3 is a free variable

Representing lines

Question: How can we describe arbitrary lines?

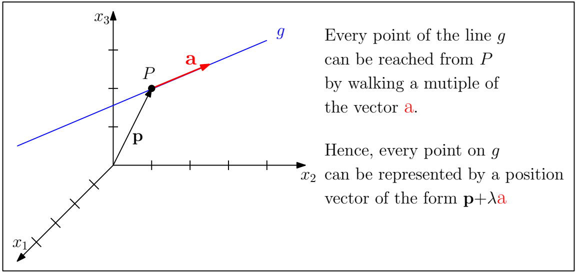
$$\begin{pmatrix}0\\3\end{pmatrix}+\operatorname{span}\left(\begin{pmatrix}2\\1\end{pmatrix}\right) = \begin{pmatrix}2\\4\end{pmatrix}+\operatorname{span}\left(\begin{pmatrix}-4\\-2\end{pmatrix}\right)$$



$$\operatorname{span}\left(\begin{pmatrix}2\\1\end{pmatrix}\right)$$

Representing lines

Idea: A line is uniquely determined if we know a point and the direction of the line.



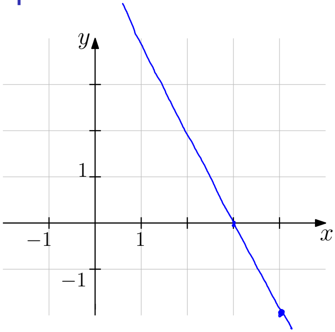
Parameter form of a line

Every line in $\mathbb{R}^2$ and $\mathbb{R}^3$ can be described in the following form:

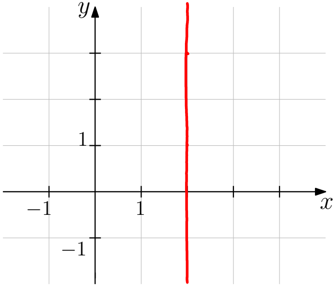
$$g = \{\mathbf{p} + \lambda\mathbf{a} : \ \lambda \in \mathbb{R}\} = \mathbf{p} + \mathsf{Span}(\mathbf{a}),$$

where the points of g are identified with their position vectors. This description is called parameter form. The vector $\mathbf{p}$ represents an arbitrary point of g, and $\mathbf{a}$ represents the direction of the line.

Example

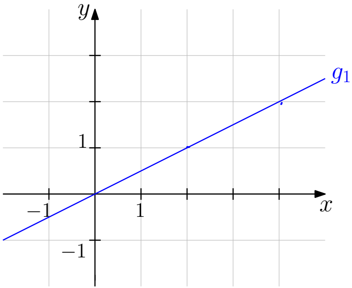


$$g_1 = \left\{ \begin{pmatrix} 4 \\ -2 \end{pmatrix} + r \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} : \; r \in \mathbb{R} \right\}$$



$$g_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \operatorname{Span}\left(\begin{pmatrix} 0 \\ -2 \end{pmatrix} \right)$$

Example

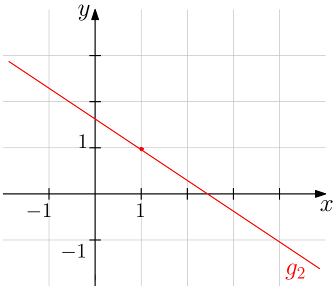


$$g_1 = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} + r \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} : r \in \mathbb{R} \right\}$$

$$= \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \operatorname{Span}\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} \right)$$

$$= \left\{ r \begin{pmatrix} 2 \\ 1 \end{pmatrix} : r \in \mathbb{R} \right\}$$

$$= \operatorname{Span}\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} \right)$$



$$g_2 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} + r \begin{pmatrix} 3 \\ -2 \end{pmatrix} : r \in \mathbb{R} \right\}$$

Exercise

(a) Describe the line g_1, which goes through the points $v_{AB} = \begin{pmatrix} -2-0 \\ 7-1 \\ 10-2 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \\ 8 \end{pmatrix}$

$$A := (0, 1, 2) \quad \text{and} \quad B := (-2, 7, 10),$$

with the help of a parameter form. $g_1 = \left\{ \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + r \begin{pmatrix} -2 \\ 6 \\ 8 \end{pmatrix} ; r \in \mathbb{R} \right\}$

(b) Describe the line g_2, which goes through the points

$$C := (5, 4, -4) \quad \text{and} \quad D := (-1, -5, -1),$$

$v_{CD} = \begin{pmatrix} -1-5 \\ -5-4 \\ -1-(-4) \end{pmatrix}$

with the help of a parameter form. $g_2 = \left\{ \begin{pmatrix} 5 \\ 4 \\ -4 \end{pmatrix} + s \cdot \begin{pmatrix} -6 \\ -9 \\ 3 \end{pmatrix} : s \in \mathbb{R} \right\}$

(c) Extra question: Do the lines g_1 and g_2 have an intersection point? If yes, determine the point.

$$\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + r \cdot \begin{pmatrix} -2 \\ 6 \\ 8 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ -4 \end{pmatrix} + s \cdot \begin{pmatrix} -6 \\ -9 \\ 3 \end{pmatrix} \Rightarrow$$

$$r \cdot \begin{pmatrix} -2 \\ 6 \\ 8 \end{pmatrix} - s \cdot \begin{pmatrix} -6 \\ -9 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ -6 \end{pmatrix}$$

$$\begin{aligned} -2 \cdot r + 6 \cdot s &= 5 \\ 6 \cdot r + 9 s &= 3 \quad | +3 \cdot \mathrm{I} \\ 8 \cdot r - 3 \cdot s &= -6 \quad | +4 \cdot \mathrm{I} \end{aligned}$$

$$\begin{aligned} -2r + 6s &= 5 \\ 0r + 27s &= 18 \Rightarrow s = \tfrac{2}{3} \\ 0r + 21s &= 14 \Rightarrow s = \tfrac{2}{3} \end{aligned}$$

Intersection point:

$$\begin{pmatrix} 5 \\ 4 \\ -4 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} -6 \\ -9 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$$

Positional relationships

g_1 is identical to g_2

g_1 is parallel to g_3

Determining positional relationships for lines

Let two lines g_1 and g_2 be given by a parameter form:

$g_1 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda_1 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} : \lambda_1 \in \mathbb{R} \right\}$

$g_2 = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \lambda_2 \cdot \begin{pmatrix} 4 \\ 2 \end{pmatrix} : \lambda_2 \in \mathbb{R} \right\}$

$$g_1 = \{\mathbf{p}_1 + \lambda_1 \mathbf{r}_1 : \ \lambda_1 \in \mathbb{R}\},$$
$$g_2 = \{\mathbf{p}_2 + \lambda_2 \mathbf{r}_2 : \ \lambda_2 \in \mathbb{R}\}.$$

$g_3 = \left\{ \begin{pmatrix} 3 \\ 3 \end{pmatrix} + \lambda_2 \cdot \begin{pmatrix} -2 \\ -1 \end{pmatrix} : \lambda_3 \in \mathbb{R} \right\}$

In order to determine the positional relationship, both representations can be equalized, resulting in a system of linear equations (with variables λ_1, λ_2):

$$\mathbf{p}_1 + \lambda_1 \mathbf{r}_1 = \mathbf{p}_2 + \lambda_2 \mathbf{r}_2$$

There are four cases:

exactly one solution	intersection point (put in λ_1, λ_2 into the parameter forms)
infinitely many solutions	identical lines
no solution (λ_1, λ_2) & $\mathbf{r}_1$ and $\mathbf{r}_2$ are multiples of each other	lines are parallel but not identical
no solution (λ_1, λ_2) & $\mathbf{r}_1$ and $\mathbf{r}_2$ are no multiples	skew lines $g_4 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda_4 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \lambda_4 \in \mathbb{R} \right\}$ $g_5 = \left\{ \begin{pmatrix} 0 \\ 8 \end{pmatrix} + \lambda_5 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} : \lambda_5 \in \mathbb{R} \right\}$