Prep Course Mathematics

Differentiation

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Derivatives of elementary functions and rules for derivatives

$f(x)$	$f'(x)$
$c \quad (c \in \mathbb{R})$	0
$x^\alpha \quad (\alpha \neq 0)$	$\alpha \cdot x^{\alpha-1}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\frac{1}{\cos^2(x)}$
$\cot(x)$	$-\frac{1}{\sin^2(x)}$
e^x	e^x
$a^x \quad (a > 0)$	$\ln(a) \cdot a^x$
$\ln(x)$	$\frac{1}{x}$
$\log_a(x) \quad (a > 0)$	$\frac{1}{\ln(a) \cdot x}$

Constant factor rule: for $c \in \mathbb{R}$

$$(c \cdot f)'(x) = c \cdot f'(x)$$

Sum rule:

$$(f \pm g)'(x) = f'(x) \pm g'(x)$$

Product rule:

$$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Quotient rule:

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

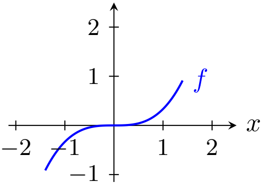
Chain rule:

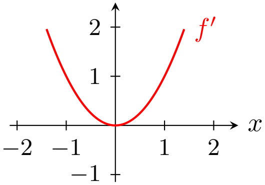
$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

Differentiability and monotonicity

For $f\colon D \to \mathbb{R}$ differentiable:

- $f'(x) \geqslant 0$ for all $x \in D \iff f$ monotonically increasing
- $f'(x) > 0$ for all $x \in D \implies f$ strictly monotonically increasing
- $f'(x) \leqslant 0$ for all $x \in D \iff f$ monotonically decreasing
- $f'(x) < 0$ for all $x \in D \implies f$ strictly monotonically decreasing





⚠ Note that $f'(x) > 0$ resp. $f'(x) < 0$ is only sufficient,
but not a necessary condition for strict monotonicity.

Example: $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) := x^3$, str. mon. increasing, but $f'(0) = 0$

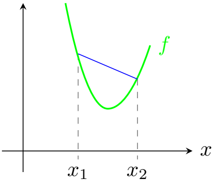
Second derivative and convexity/concavity

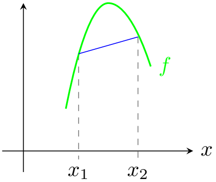
For D interval, $f\colon D \to \mathbb{R}$:

- f convex, if for all $x_1, x_2 \in D$ and $\lambda \in (0,1)$:

$$f(\lambda x_1 + (1-\lambda)x_2) \leqslant \lambda f(x_1) + (1-\lambda) f(x_2).$$

- f strictly convex if "$<$" instead of "$\leqslant$"
- f (strictly) concave, if $-f$ (strictly) convex,
  i.e. "$\geqslant$" ("$>$") instead of "$\leqslant$" ("$<$")





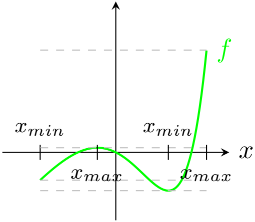
For f two times differentiable:

- $f''(x) \geqslant 0$ for all $x \in D \iff f$ convex.
- $f''(x) > 0$ for all $x \in D \implies f$ strictly convex.
- $f''(x) \leqslant 0$ for all $x \in D \iff f$ concave.
- $f''(x) < 0$ for all $x \in D \implies f$ strictly concave.

Local extrema

For $D \subset \mathbb{R}$ and function $f\colon D \to \mathbb{R}$:

$x_0 \in D$ is called



- local maximum if there exists a neighbourhood U around x_0 such that $f(x) \leqslant f(x_0)$ for $x \in U$
- local minimum if there exists a neighbourhood U around x_0 such that $f(x) \geqslant f(x_0)$ for $x \in U$
- local extremum if it is a local maximum or a local minimum.
- global maximum if $f(x) \leqslant f(x_0)$ for all $x \in D$.
- global minimum if $f(x) \geqslant f(x_0)$ for all $x \in D$.

Neighbourhood around U = (arbitrarily small) open subinterval of D that contains x_0.

Theorem

For $f\colon D \to \mathbb{R}$ differentiable, $x_0 \in D$ interior point:
If f has a local extremum at x_0, then $f'(x_0) = 0$.

⚠ For local extrema at boundary points x_0 we can have $f'(x_0) \neq 0$.

$x_0 \in D$ stationary point if $f'(x_0) = 0$.

Criteria for extrema

For $f\colon (a,b) \to \mathbb{R}$ differentiable, $x_0 \in (a,b)$ stationary point:

…using the first derivative:

Theorem

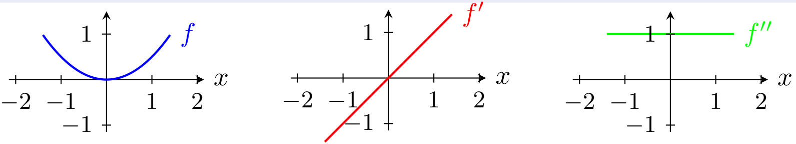
- *If $f'(x) \geqslant 0$ for $x < x_0$ and $f'(x) \leqslant 0$ for $x > x_0$, then f has a local maximum in x_0.*
- *If $f'(x) \leqslant 0$ for $x < x_0$ and $f'(x) \geqslant 0$ for $x > x_0$, then f has a local minimum in x_0.*

…using the second derivative:

Theorem

For f two times differentiable:

- *If $f''(x_0) < 0$, then f has a local maximum in x_0.*
- *If $f''(x_0) > 0$, then f has a local minimum in x_0.*



Inflection points

For $f\colon (a,b) \to \mathbb{R}$, $x_0 \in (a,b)$:
f has inflection point at x_0, if the second derivative changes its sign.

Theorem (Criteria for inflection points)

- *If f is two times differentiable and has an inflection point at x_0, then $f''(x_0) = 0$.*
- *If f is three times differentiable, $f''(x_0) = 0$ and $f'''(x_0) \neq 0$, then f has an inflection point at x_0.*