Prep Course Mathematics

Integration

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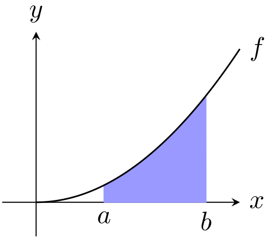
Motivation

Given $f\colon \mathbb{R} \to \mathbb{R}$.

Integral calculus: Area under f in $[a, b]$



Integral

$$\int_a^b f(x)\,\mathrm{d}x$$

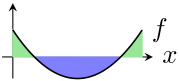
- a, b limits (or bounds) of integration
- f integrand
- x variable of integration
- $\int_a^b f(x)\,\mathrm{d}x = \int_a^b f(t)\,\mathrm{d}t = \int_a^b f(€)\,\mathrm{d}€$

Linearity

- For $\alpha, \beta \in \mathbb{R}$: $\int_a^b (\alpha f + \beta g)(x)\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x + \beta \int_a^b g(x)\,\mathrm{d}x$

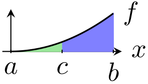
Positivity and monotonicity:

- $f \geqslant 0 \implies \int_a^b f(x)\,\mathrm{d}x \geqslant 0$
- $f \leqslant g \implies \int_a^b f(x)\,\mathrm{d}x \leqslant \int_a^b g(x)\,\mathrm{d}x$
- $\int_a^b f(x)\,\mathrm{d}x =$ area where $f \geqslant 0$ $-$ area where $f < 0$



Partition and absolute value:

- For $a < c < b$: $\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$
- $\int_b^a f(x)\,\mathrm{d}x := -\int_a^b f(x)\,\mathrm{d}x$, and $\int_a^a f(x)\,\mathrm{d}x = 0$
- $\left|\int_a^b f(x)\,\mathrm{d}x\right| \leqslant \int_a^b |f(x)|\,\mathrm{d}x$



Antiderivative

F is an antiderivative of f if F is differentiable and $F' = f$

Example:

Let $f : [0,1] \to \mathbb{R}$ with $f(x) := 2x$

Antiderivative: $F : [0,1] \to \mathbb{R}$ with $F(x) = x^2$

- If F is an antiderivative of f, then also $F + c$ with any real number c.
- Antiderivatives are unique up to a constant.
- "Antiderivatives form the reversal of derivatives"
- If F is an antiderivative, then it holds

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

Hence, the calculation of integrals is reduced to the calculation of antiderivatives and their evaluation at the integral limits.

Determination of antiderivatives

"Differentiation is mechanics, integration is art."

$f(x)$	$f'(x)$
x^α	$\alpha x^{\alpha-1}$
e^x	e^x
$\ln\|x\|$	$\frac{1}{x}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\int f(x)\,\mathrm{d}x + c$	$f(x)$

linearity:

$$\int \alpha f(x)+\beta g(x)\,\mathrm{d}x = \alpha \int f(x)\,\mathrm{d}x + \beta \int g(x)\,\mathrm{d}x$$

products: ➪ integration by parts

$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x$$

(special) quotients:
➪ partial fraction decomposition

(special) compositions: ➪ substitution

$$\int f(g(x))g'(x)\,\mathrm{d}x = F(g(x)) + c$$

Examples

① $f(x) = \frac{1}{x^2} = x^{-2}$

$F(x) = \frac{1}{-2+1} x^{-2+1} = \frac{1}{-1} x^{-1} = -x^{-1} = -\frac{1}{x}$

② $f(x) = 2\sin(x) - 3\cos(x) + 4x$

$F(x) = 2(-\cos(x)) - 3\sin(x) + 4 \cdot \frac{1}{1+1} x^{1+1} = -2\cos(x) - 3\sin(x) + 2x^2$

③ $f(x) = x^{-2} - \cos(x) - 3e^x$

$F(x) = -\frac{1}{x} - \sin(x) - 3e^x$

Indefinite integral

The integration limits aren't necessary

We get a set of antiderivatives as the result

Examples:

1. $\int \left(\frac{2}{x^2} - 6x^2 \right) dx = -\frac{2}{x} - 2x^3 + c$

2. $\int (a\sin(x) + b\cos(x)) dx = -a\cos(x) + b\sin(x) + c$

3. $\int \frac{ax+b}{x^3} dx = \int \left(\frac{a}{x^2} + \frac{b}{x^3} \right) dx = -\frac{a}{x} - \frac{b}{2x^2} + c$

4. $\int \left(\frac{1}{(ax)^6} - b^2 e^x \right) dx = \int \left(\underbrace{\frac{1}{a^6 x^6}}_{\frac{1}{a^6} \cdot x^{-6}} - b^2 e^x \right) dx = -\frac{1}{5a^6 x^5} - b^2 e^x + c$

$\frac{1}{a^6} \cdot x^{-6} \xrightarrow{\int \ldots} \frac{1}{a^6} \frac{1}{-5} x^{-5}$

(5) $\int 5x^2\sqrt[3]{x}\,dx = \int 5x^2 \cdot x^{\frac{1}{3}}\,dx = \int 5x^{\frac{7}{3}}\,dx = \frac{5}{\frac{10}{3}} x^{\frac{10}{3}} + c = \frac{3}{2} x^{\frac{10}{3}} + c$

(6) $\int \frac{(x-3)^2}{\sqrt{x}}\,dx = \int \frac{x^2 - 6x + 9}{x^{1/2}}\,dx = \int \left(\frac{x^2}{x^{\frac{1}{2}}} - \frac{6x}{x^{\frac{1}{2}}} + \frac{9}{x^{\frac{1}{2}}} \right) dx = \int \left(x^{\frac{3}{2}} - 6x^{\frac{1}{2}} + 9x^{-\frac{1}{2}} \right) dx$

$= \frac{2}{5} x^{\frac{5}{2}} - 4x^{\frac{3}{2}} + 18x^{\frac{1}{2}} + c$

(7) $\int \frac{e^x + x^4}{7}\,dx = \frac{1}{7} \int (e^x + x^4)\,dx = \frac{1}{7}\left(e^x + \frac{1}{5}x^5 \right) + c$

Calculation of integrals

(1) $\int_1^2 \left(\frac{2}{x^2} - 6x^2 \right) dx = \left[-\frac{2}{x} - 2x^3 \right]_1^2 = \left(-\frac{2}{2} - 2 \cdot 2^3 \right) - \left(-\frac{2}{1} - 2 \cdot 1^3 \right) = (-1 - 16) - (-2 - 2)$
$= -17 + 4 = -13$

(2) $\int_{-2}^{-1} \frac{1+x}{x^2}\,dx = \int_{-2}^{-1} \left(\frac{1}{x^2} + \frac{1}{x} \right) dx = \left[-\frac{1}{x} + \ln|x| \right]_{-2}^{-1} = \left(\frac{1}{1} + \ln(1) \right) - \left(\frac{1}{2} + \ln(2) \right) = \frac{1}{2} - \ln(2)$

(3) $\int_{\pi}^{0} (2\sin(x) - 3e^x)\,dx = \left[-2\cos(x) - 3e^x \right]_{\pi}^{0} = (-2 \cdot 1 - 3 \cdot 1) - (-2 \cdot (-1) - 3e^{\pi})$
$= (-2 - 3) - (2 - 3e^{\pi}) = 3e^{\pi} - 7$