Prep Course Mathematics

Proofs

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Content

1. Methods of proof
   - Direct proof
   - Indirect proof
   - Mathematical induction

What is a proof?

Definition (Proof)

logical list of arguments, starting from a given assumption to verify (or falsify) an assertion.

⚠ As long as a statement is not proven, it may be that it is false.

Example: $F_n = 2^{2^n} + 1, \quad n \in \mathbb{N}_0$

Conjecture of Fermat (1637): all F_n are prime number.
Disproved from Euler (1732): He found 641 a real divisor of

$$F_5 = 4.294.967.297.$$

Approach:

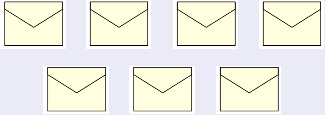
1. Understand the question: know the relevant definitions
2. Choose method of proof: similar questions known?
3. Perform the proof
4. Check: question answered, all intermediate steps correct?

Proof: When is an example enough and when not?

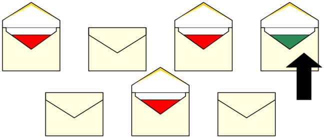
	prove	**disprove**
It-exists-statement $\exists x : A(x)$	**?**	**?**
For-all-statement $\forall x : A(x)$	**?**	**?**

Example: Prove it-exists-statement

Among the following letters
there exists one with a green card.



Proof:



We found a letter with a green card.
So the statement is proven.
It doesn't matter if there are other such letters!

Example:

There exist sets of natural numbers with infinitely many elements but not containing a common element.

Take all odd numbers and all even numbers

$A = \{1, 3, 5, 7, \dots\}$

$B = \{2, 4, 6, 8, \dots\}$

Thus the statement is true.

q.e.d.

$\square$

$A = \{3^1, 3^2, 3^3, 3^4, \dots\}$

$B = \{5^1, 5^2, 5^3, 5^4, \dots\}$

Proof: When is an example enough and when not?

1. Case: Prove it-exists-statement

Suppose we have a statement of the form

„It exists an object x that fulfils $A(x)$ "

✎ $\exists x : A(x)$

To prove such a statement, an **example** is enough

Reason: The statement only calls for one object,
which has the desired property $A(x)$

⚠ **Attention:**

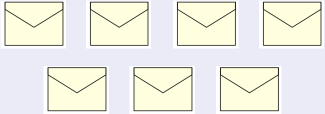
If we say ,,It exists ...",
then we mean: ,,It exists at least one ...".
So there could be two, three or more.

Proof: When is an example enough and when not?

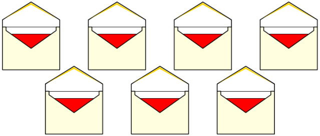
	prove	**disprove**
It-exists-statement $\exists x : A(x)$	**example**: Show that an x has the property $A(x)$.	**?**
For-all-statement $\forall x : A(x)$	**?**	**?**

Example: Prove for-all-statement

All of the following letters have a red card.



Proof:



Only when you know for each letter that there is a red card, the statement is proven! Just opening a few letters is not enough.

Example: For all $x \in \mathbb{R}$ it holds that $x^2 - 6x + 10 \geq 0$

$$\underbrace{1}_{\geq 0} + \underbrace{(x-3)^2}_{\geq 0} = x^2 - 6x + 9 + 1 = x^2 - 6x + 10$$

Thus we have the sum of two summands which are always at least 0, so our statement is true for all $x \in \mathbb{R}$.

Proof: When is an example enough and when not?

2. Case: Prove for-all-statement

Suppose we have a statement of the form

„All objects x fulfil $A(x)$ “

✎ $\forall x : A(x)$

To prove such a statement, an example is NOT enough.
A **generally valid proof** is necessary!

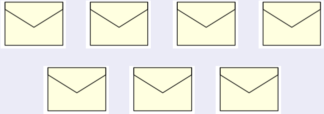
Reason: To know that an object has the property $A(x)$
does not mean that all objects have this property

Proof: When is an example enough and when not?

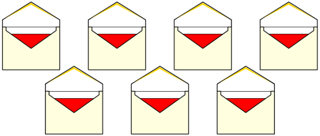
	prove	**disprove**
It-exists-statement $\exists x : A(x)$	**example**: Show that an x has the property $A(x)$.	**?**
For-all-statement $\forall x : A(x)$	**generally valid proof**: Show that all x have the property $A(x)$.	**?**

Example: Disprove it-exists-statement

Among the following letters
there exists one with a green card.



Proof:



Only when you know for each letter that there is a red card, the statement is disproved! Just opening a few letters is not enough.

Proof: When is an example enough and when not?

3. Case: Disprove it-exists-statement

Suppose we have a statement of the form

„It exists an object x that fulfils $A(x)$ "

✎ $\exists x : A(x)$

To disprove such a statement means to **prove the opposite**.
Reason: Either a statement or its opposite is true.

The opposite is a *for-all-statement*:

„All objects x do not fulfil $A(x)$ "

✎ $\forall x : \neg A(x)$

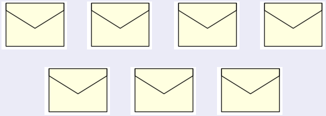
To prove this a **generally valid proof** is necessary!

Proof: When is an example enough and when not?

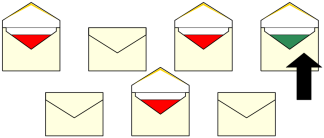
	prove	**disprove**
It-exists-statement $\exists x : A(x)$	**example**: Show that an x has the property $A(x)$.	**generally valid proof**: Show that an x do $A(x)$ not have the property $A(x)$.
For-all-statement $\forall x : A(x)$	**generally valid proof**: Show that all x have the property $A(x)$.	**?**

Example: Disprove for-all-statement

All of the following letters have a red card.



Proof:



We found a letter with a green card. So the statement is disproven.
It doesn't matter if there are other such letters!

Example:

For all $n \in \mathbb{N}$ n^2+n+41 is a prime number.

$B(x)$: x is a prime number.

A: $\forall n \in \mathbb{N}: B(x)$

Negation:

$\neg A$: $\exists n \in \mathbb{N}: \neg B(x)$ There exists $n \in \mathbb{N}$ such that n^2+n+41 is not prime.

$n=41$: $41^2+41+41 = 41 \cdot 41 + 41 + 41 = 41 \cdot (41+1+1) = 41 \cdot 43$

this is not a prime number!

Thus $\neg A$ is true and A is false.

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a) $\forall x \in \mathbb{R}: (x+4)(x+7) \geq 0$ false for $x=-5$: $-1 \cdot 2 = -2 \not\geq 0$

Thus the statement is false.

b) $\forall z \in \mathbb{Z}: (z-\frac{1}{2})(z-\frac{2}{3}) > 0$

$z > \frac{2}{3}$: $(z-\frac{1}{2}) > 0$ and $(z-\frac{2}{3}) > 0$

$z < \frac{1}{2}$: $(z-\frac{1}{2}) < 0$ and $(z-\frac{2}{3}) < 0$

Thus the statement is true for all $z \in \mathbb{Z}$.

Proof: When is an example enough and when not?

4. Case: Disprove for-all-statements

Suppose we have a statement of the form

„All objects x fulfil $A(x)$ "

✎ $\forall x : A(x)$

To disprove such a statement means to **prove the opposite**.
Reason: Either a statement or its opposite is true.

The opposite is an for-all-statement:

„It exists an object x that does not fulfil $A(x)$. "

✎ $\exists x : \neg A(x)$

A **(counter-)example** is enough.

Proof: When is an example enough and when not?

	prove	**disprove**
It-exists-statement $\exists x : A(x)$	**example**: Show that an x has the property $A(x)$.	**generally valid proof**: Show that an x do $A(x)$ not have the property $A(x)$.
For-all-statement $\forall x : A(x)$	**generally valid proof**: Show that all x have the property $A(x)$.	**counter example**: Show that an x does not have the property $A(x)$.

Examples

There exist natural number a, b, and c, such that $a^2 + b^2 = c^2$ holds.

Proof: (Proving It-exists-statement with an example)
For example consider $a = 3$, $b = 4$, and $c = 5$.
Then $a^2 + b^2 = 9 + 16 = 25 = c^2$.

For every real number x it holds that $x^2 - 8x + 17 \geqslant 0$.

Observation:
For $x = 1$ we have $1^2 - 8 \cdot 1 + 17 = 10 \geqslant 0$.✓
For $x = 2$ we have $2^2 - 8 \cdot 2 + 17 = 5 \geqslant 0$.✓
For $x = 3$ we have $3^2 - 8 \cdot 3 + 17 = 2 \geqslant 0$.✓
But why does the inequality hold for all $x \in \mathbb{R}$?

Proof: (Proving For-all-statement with a generally valid proof.)

Let x be a real number. Then the following holds:
$x^2 - 8x + 17 = (x - 4)^2 + 1$.
This statement is always at least 0, since both the square $(x - 4)^2$ and the summand 1 are non-negative.

Methods of proof

Direct proof

- Given: A
- Find: B
- Show that $A \implies B$, usually via
  $A \implies A_1 \implies A_2 \implies \ldots \implies A_n \implies B$.

Indirect proof via contraposition

- Given: A
- Find: B
- Show that $A \implies B$, by showing $\neg B \implies \neg A$.

Indirect proof via contradiction

- Show that A, by falsifying $\neg A$.

Mathematical induction

Example direct proof.

If in a right angled triangle the shorter sides are of length 3cm and 4cm, then the longest side is 5cm long.

Proof: Since we have a right angled triangle, we can apply Pythagoras:

$$c^2 = a^2 + b^2 \Rightarrow c = \sqrt{a^2+b^2} = \sqrt{(3cm)^2 + (4cm)^2} = \dots = 5cm \qquad \square$$

Example indirect proof via contraposition:

Let $n \in \mathbb{N}$. If $\underbrace{n^2 \text{ is even}}_{A}$, then $\underbrace{n \text{ is even}}_{B}$.

Instead of proving $A \Rightarrow B$ we show that $\neg B \Rightarrow \neg A$.

Proof: Let $\overset{\neg B}{n \text{ be odd}}$. Then there exists an integer m with $n = 2m-1$

$$\Rightarrow n^2 = (2m-1)^2 = 4m^2 - 4m + 1 = 2 \cdot \underbrace{(2m^2 - 2m)}_{z} + 1$$

$\Rightarrow$ there exists an integer z with $n^2 = 2 \cdot z + 1$

$\Rightarrow \underbrace{n^2 \text{ is odd}}_{\neg A}$ Thus our original statement is true. $\square$

Example: Indirect proof, contradiction

A: $\sqrt{2}$ is not a rational number.

$\neg A$. $\sqrt{2}$ is a rational number.

Under the assumption that $\sqrt{2}$ is rational, there exist $n, m \in \mathbb{N}$ such that

$\sqrt{2} = \frac{m}{n}$ and n and m have no common factor.

$\overset{(\dots)^2}{\Rightarrow}$ $2 = \frac{m^2}{n^2}$ $\Rightarrow$ $2n^2 = m^2$ $\Rightarrow$ m^2 is even $\Rightarrow$ m is even

$\Rightarrow$ there exists $k \in \mathbb{N}$ with $m = 2k$

$\Rightarrow$ $2n^2 = m^2 = (2k)^2 = 4k^2$ $\Rightarrow$ $n^2 = 2k^2$ $\Rightarrow$ n^2 is even $\Rightarrow$ n is even

$\Rightarrow$ m and n are divisible by 2, which is a contradiction to the assumption that m and n have no common factor. $\lightning$

$\Rightarrow$ $\neg A$ is false and A has to be true.

$\Rightarrow$ $\sqrt{2}$ is not a rational number.

$\square$

q.e.d.

Mathematical induction

Aim: A predicate $A(n)$ should be proved for all natural numbers $n \geqslant n_0$, where $n_0 \in \mathbb{N}$.

Mathematical induction

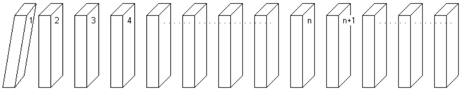
To show that the predicate $A(n)$ is true for all $n \geqslant n_0$, can be proved as follows:

- **Base case:** Show that $A(n_0)$ is true.
- **Induction step:** Show that $A(n+1)$ is true
  under the assumption that $A(n)$ is true for some $n \geqslant n_0$.

Short: $A(n) \Rightarrow A(n+1)$

$A(n)$ is called the induction hypothesis

Domino effect



Example induction:

we have a recursively defined sequence $a_1, a_2, a_3, \dots$

$a_1 = 1$

and

$a_{n+1} = a_n + 2n + 1$

$a_1 = 1$

$a_2 = a_1 + 2\cdot 1 + 1 = 1 + 2 + 1 = 4$

$a_3 = a_2 + 2\cdot 2 + 1 = 4 + 4 + 1 = 9$

$a_4 = a_3 + 2\cdot 3 + 1 = 9 + 6 + 1 = 16$

Show than $a_n = n^2$. $\forall n \in \mathbb{N}$.

$A(1)$: $a_1 = 1^2 = 1$

$A(n)$: $a_n = n^2$

$A(n+1)$: $a_{n+1} = (n+1)^2$

Proof via induction:

Base case: $n = 1$, show that $A(1)$ holds. $a_1 = 1 = 1^2$ ✓

Induction step: $k \to k+1$, show that $A(k) \Rightarrow A(k+1)$ $\forall k \in \mathbb{N}$

We can assume that $a_k = k^2$

$a_{k+1} = a_k + 2k + 1 = k^2 + 2k + 1 = (k+1)^2$ ✓

Thus the statement is true for all $n \in \mathbb{N}$. □